A study of radiative-conductive heat transfer in spherical enclosure using discrete ordinates method associated to ultraspherical polynomials approximation

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Résumé – Une nouvelle méthode de résolution de l'équation de transfert radiatif dans un milieu semi transparent sphérique est développée. Cette approche est basée sur l'association de la méthode des ordonnées discrètes et une projection sur une base des polynômes ultrasphériques. Cette projection est utilisée pour l'approximation du terme de dérivée angulaire. Nous avons appliqué la nouvelle approche à un problème de transfert couplé conduction –rayonnement. Les résultats obtenus sont en bon accord avec ceux de la bibliographie.

1. Introduction

In practical engineering problems, simultaneous conduction and radiation in participating media appears in many applications such as combusting systems, furnaces and reactor nuclear theory. Many of these systems can be considered as spherical enclosure. A number of studies interested in resolving the radiative transfer equation in such media have been conducted. These works included various numerical techniques: integral transformation techniques [1], spherical harmonics method [2], Galerkin method [3]. In this paper, simultaneous conduction and radiation in one-dimensional absorbing, emitting and isotropically scattering gray hollow sphere with gray surfaces is investigated. We introduce a new technique for improving the performance of the discrete ordinates method. The novelty of this technique lies in the use of ultraspherical polynomials approximations [4] to represent the angular derivative term of the discretized 1-D radiative transfer equation. The set of resulting differential equations is solved using the boundary value problem with finite difference algorithm [5].

2. Analysis

We consider one-dimensional steady-state simultaneous conduction and radiation in gray absorbing emitting and isotropically scattering hollow sphere. The limiting surfaces are assumed to be gray, opaque, diffusely emitting, diffusely reflecting and the physical properties are constant. The energy equation is taken

$$k\left(\frac{d^2T}{dr^2}+\frac{2}{r}\frac{dT}{dr}\right)=\frac{dq_r}{dr}+\frac{2}{r}q_r \text{ in } R_1<r<R_2 \tag{1}$$

with boundary conditions,

$$T(R_1)=T_1\ ,T(R_2)=T_2 \tag{2}$$

Where T is the temperature, k the thermal conductivity and q_r the radiation flux.

The dimensionless formulation of the problem defined by Eqs. (1) can be written as :

$$\frac{d^2\Theta}{dr^{*2}} + \frac{2}{r^*}\frac{d\Theta}{dr^*} = \frac{\tau_2}{4N_{cr}}\left(\frac{dq_r^*}{dr^*} + \frac{2}{r^*}q_r^*\right), \quad R_1^* < r^* < 1, \tag{3}$$

$$\Theta(R_1^*) = 1, \quad \Theta(1) = \Theta_2 \tag{4}$$

The divergence of dimensionless heat flux vector in Eq. (2) can be expressed as

$$\frac{dq_r^*}{dr^*} + \frac{2}{r^*}q_r^* = 4(1-\omega)\tau_2(\Theta^4 - G^*) \tag{5}$$

and the dimensionless incident radiation is defined as

$$G^* = \pi G / n^2 \sigma T_1^4 . \tag{6}$$

Substituting Eq. (3) into Eq. (2), we obtain

$$\frac{d^2\Theta}{dr^{*2}} + \frac{2}{r^*}\frac{d\Theta}{dr^*} = \frac{(1-\omega)}{N_{cr}}\tau_2^2(\Theta^4 - G^*). \tag{7}$$

Where τ_2 is the optical thickness and ω the single scattering albedo. n is the refractive index, σ the Stefan Boltzmann constant and N_{cr} the conduction radiation parameter.

The dimensionless radiation flux q_r^* and the dimensionless incident radiation G^* must be determined from the solution of the radiative transfer equation (RTE), in spherical medium, given by

$$\frac{\mu}{r^{*2}}\frac{\partial(r^{*2}\psi)}{\partial r^*} + \frac{1}{r^*}\frac{\partial[(1-\mu^2)]\psi}{\partial\mu} + \tau_2\psi = \tau_2(1-\omega)\Theta^4 + \tau_2\frac{\omega}{2}\int_{-1}^{+1}\psi(r^*,\mu')d\mu' \tag{8}$$

$$-1 < \mu < +1, R_1^* < r^* < R_2^*$$

with the diffuse boundary conditions

$$\psi(R_1^*,\mu) = \varepsilon_1\Theta_1^4 + 2(1-\varepsilon_1)\int_0^1 \psi(R_1^*,-\mu')|\mu'|d\mu', \qquad \mu > 0, \tag{9a}$$

$$\psi(R_2^*,\mu) = \varepsilon_2\Theta_2^4 + 2(1-\varepsilon_2)\int_0^1 \psi(R_2^*,-\mu')\mu' d\mu', \qquad \mu < 0. \tag{9b}$$

Where μ is the cosine of the angle between the direction of the radiation intensity ψ and the positive r^* axis. ε is the surface emissivity. The subscripts 1 and 2 refer to the boundaries at $r^* = R_1^*$ and $r^* = R_2^*$ respectively. The discrete form of the radiative transfer equation is obtained by evaluating Eq. (15) at each of the discrete directions and replacing the integral by numerical quadrature to give

$$\frac{\mu_m}{r^{*2}}\frac{\partial(r^{*2}\psi_m)}{\partial r^*} + \frac{1}{r^*}\frac{\partial[(1-\mu^2)]\psi_m}{\partial\mu} + \tau_2\psi = \tau_2(1-\omega)\Theta^4 + \tau_2\frac{\omega}{2}\sum_{m'=1}^{M} w_{m'}\psi_{m'} \tag{10}$$

Where subscripts m and m' refer to discrete directions, M is the total number of these directions. The discrete ordinate representation of the boundary conditions, Eqs. (9a)-(9b) is given by

$$\psi_m(R_1^*) = \varepsilon_1\Theta_1^4 + 2(1-\varepsilon_1)\sum_{m'=1,\ \mu_{m'},<0}^{M} w_{m'}\psi_{m'}\left|\mu_{m'}\right| \qquad \mu_m > 0 \tag{11a}$$

$$\psi_m(R_2^*) = \varepsilon_2\Theta_2^4 + 2(1-\varepsilon_2)\sum_{m'=1,\ \mu_{m'},>0}^{M} w_{m'}\psi_{m'}\mu_{m'} \qquad \mu_m < 0\,. \tag{11b}$$

The discrete form of the term involving the angular derivation in Eq. (10) is generally expressed using a finite differencing scheme [6]. We develop in the following an alternative method for the angular derivative term.

3. Ultraspherical P_N^λ method

In this section we develop a new approach to evaluate the angular derivative term given by

$$ADT = \frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\psi\right] \tag{12}$$

The nth ultraspherical moment of the *ADT*, denoted by $D(r,\mu)$, can be defined as follows

$$\phi_n(r) = \int_{-1}^{+1} D(r,\mu)P_n^\lambda(\mu)d\mu \tag{13}$$

where P_n^λ is the ultraspherical polynomial of order n and variable λ. Legendre polynomial (P_n), Chebyshev polynomials of first and second kind (T_n ,U_n) are special cases of ultra spherical or Gegenbauer polynomials. The integral In Eq. (9) could be simplified using integration by parts as follows

$$\int_{-1}^{+1} D(r,\mu)P_n^\lambda(\mu)d\mu = -\int_{-1}^{+1}\psi(r,\mu)(1-\mu^2)\frac{dP_n^\lambda}{d\mu}d\mu \tag{14}$$

Using the equation defining the derivative of ultraspherical polynomials given by

$$(1-\mu^2)\frac{dP_n^\lambda}{d\mu} = \frac{1}{2(n+\lambda)}\left[(n+2\lambda-1)(n+2\lambda)P_{n-1}^\lambda - n(n+1)P_{n+1}^\lambda\right], \tag{15}$$

then, Eq. (10) may be written as

$$\begin{aligned}\int_{-1}^{+1} D(r,\mu)P_n^\lambda(\mu)d\mu = \frac{n(n+1)}{2(n+\lambda)}\int_{-1}^{+1}\psi(r,\mu)P_{n+1}^\lambda d\mu -\\ \frac{(n+2\lambda-1)(n+2\lambda)}{2(n+\lambda)}\int_{-1}^{+1}\psi(r,\mu)P_{n-1}^\lambda d\mu\end{aligned} \tag{16}$$

In order to obtain the *ADT*, the integral equation given by Eq. (12) is replaced by the discrete form

$$\sum_{m=1}^{M} w_m D_m P_n^{\lambda}(\mu_m) = \frac{n(n+1)}{2(n+\lambda)} \sum_{m=1}^{M} w_m \psi_m P_{n+1}^{\lambda}(\mu_m) - \frac{(n+2\lambda-1)(n+2\lambda)}{2(n+\lambda)} \sum_{m=1}^{M} w_m \psi_m P_{n-1}^{\lambda}(\mu_m) \tag{17}$$

Where $D_m = \partial / \partial\mu \left[\left(1-\mu^2 \right) \psi \right]_{\mu=\mu_m}$, w_m are the quadrature weights associated with the directions μ_m and M is the total number of discrete directions. For one-dimensional spherical problem using discrete ordinates method or S_N method, the value of M is equal to the order of S_N approximation. Now the angular derivative terms D_m are obtained following the procedure described by Sghaier et al. [7]

$$D_m = \frac{1}{w_m} \sum_{j=1}^{M} (A^{-1})_{mj} B_j , \tag{18}$$

where

$$B_j = \frac{j(j+1)}{2(j+\lambda)} \sum_{m'=1}^{M} w_{m'} \psi_{m'} P_{j+1,m'}^{\lambda} - \frac{(j-1+2\lambda)(j+2\lambda)}{2(j+\lambda)} \sum_{m'=1}^{M} w_{m'} \psi_{m'} P_{j-1,m'}^{\lambda} , \boldsymbol{j \geq 1} ,$$

and A^{-1} is the inverse of the matrix A which is given by $A = \left(P_{mj}^{\lambda} \right)_{\substack{0 \leq j \leq M-1 \\ 1 \leq m \leq M}}$. It is called the ultraspherical-Vandermonde matrix. The new representation of Eq. (8), for a finite number of discrete ordinates may be written as

$$\frac{\mu_m}{\tau_2 r^{*2}} \frac{\partial}{\partial r^*} (r^{*2} \psi_m) + \frac{1}{\tau_2 r^*} \frac{1}{w_m} \sum_{j=1}^{M} (A^{-1})_{mj} B_j + \psi_m = (1-\omega)\Theta^4 \left(r^* \right) + \frac{\omega}{2} \sum_{m'=1}^{M} w_{m'} \psi_{m'} , \ m = 1,M \tag{19}$$

Finally the dimensionless conductive heat flux Q_c, the radiation heat flux q_r^* and total heat flux Q_T are determined from the following relations

$$Q_c(r^*) = -\frac{d\Theta}{dr^*} ; q_r^* = 2\sum_{m=1}^{M} \mu_m w_m \psi_m ; Q_T = -\frac{d\Theta}{dr^*} + \frac{\tau_2}{4N_{cr}} \sum_{m=1}^{M} \mu_m w_m \psi_m \tag{20}$$

Equations (7), (19) and (11a, 11b) provide the complete mathematical formulation for the conduction-radiation problem in a one dimensional spherical medium. A numerical technique, namely the boundary value problem with finite difference (BVPFD) [5] is used to solve this problem. The new technique called $P_N^{\lambda} - DOM$ method with N=12 is adopted for different pre-selected values of λ. Each value of λ leads to a different approximation. The weights and quadrature points are those of the corresponding Gaussian quadratures.

4. Results and Discussion

Method	*DOM* [8]	$T_N - DOM$ [8] $\lambda = 0$	$P_N - DOM$ $\lambda = 0.5$	$U_N - DOM$ $\lambda = 1$	$P_N^2 - DOM$ $\lambda = 2$	*Galerkin* [3]
$Q_c(r_1^*)$	2.2978	2.3022	2.2986	2.2985	2.2985	2.3223
$Q_c(r_2^*)$	0.6464	0.6457	0.6464	0.6464	0.6464	0.6426
$Q_T(r_1^*)$	6.0258	5.9862	5.9945	5.9945	5.9946	5.9433
$Q_T(r_2^*)$	1.5064	1.4966	1.4987	1.4986	1.4987	1.4858

Table 1: Values of the dimensionless conductive heat flux and total heat flux at the boundaries for different approximations.

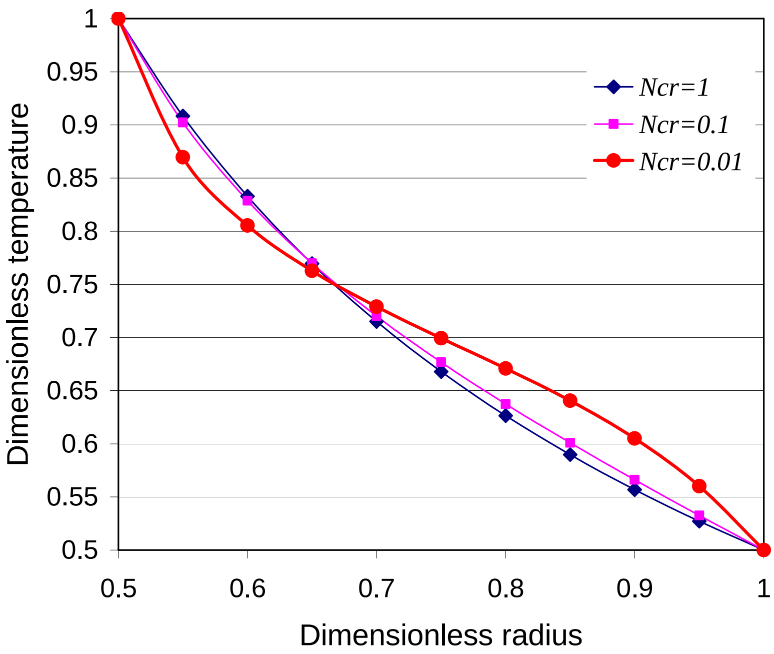

Figure 1 : The effect of the conduction –radiation parameter on the dimensionless temperature profile

In table 1 numerical values of the dimensionless conductive heat flux and total heat flux at the boundaries for different approximations are presented. Results obtained by the standard discrete ordinates method denoted by *DOM* in the table and from new formulation by P_N^{λ} – *DOM* method are compared with the results of Jia et al [3], which are obtained by Galerkin method. We present results for $R_1 / R_2 = 0.5, \tau_2 = 2$, $\varepsilon_1 = \varepsilon_2 = 1, \Theta_2 = 0.5$. It can be observed from table 1 that all P_N^{λ} –*DOM* results are consistent in themselves equiconvergent and in good agreement with the comparable data.

An analysis of simultaneous conduction and radiation in one dimensional, absorbing, emitting and isotropically scattering hollow spherical medium is investigated. The angular derivative term appearing in this geometry is approximated by making use of a new approach called $P_N^{\lambda} - DOM$ approximation. This method leads to an accurate expression for the angular derivative term. The set of resulting differential equations is solved using the boundary value problem with finite difference algorithm. The solution accuracy of $P_N^{\lambda} - DOM$ method has been verified by comparison with benchmark approximate solutions.

The effects of the conduction radiation parameter N_{cr} on the dimensionless temperature distribution are studied in Fig. 1 using $U_N - DOM$ approximation. Calculations are carried out for $\tau_2 = 2$; $\omega = 0.5$; $R_1 / R_2 = 0.5; \Theta_2 = 0.5$ and $\varepsilon_1 = \varepsilon_2 = 1$. As N_{cr} decreases, radiation plays more significant role than conduction. Therefore, as N_{cr} decreases, a steeper temperature gradient is formed at both boundaries as shown in Fig. 1. Let us note that this new formulation does not present any new difficulty as compared to classical formulation. The matrix A depends only on the chosen quadrature and on some values of Gegenbauer polynomials; its inversion is to be carried out once only.

5. Conclusion

An analysis of simultaneous conduction and radiation in one dimensional, absorbing, emitting and isotropically scattering hollow spherical medium is investigated. The angular derivative term appearing in this geometry is approximated by making use of a new approach called $P_N^{\lambda} - DOM$ approximation. This method leads to an accurate expression for the angular derivative term. The set of resulting differential equations is solved using the boundary value problem with finite difference algorithm. The solution accuracy of $P_N^{\lambda} - DOM$ method has been verified by comparison with benchmark approximate solutions.

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