Lecure 3: Models and measurements for thermal systems
Types of inverse problems

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1. Objectives, models & direct problems, internal/external representations

2. Parameterizing a function & parcimony principle

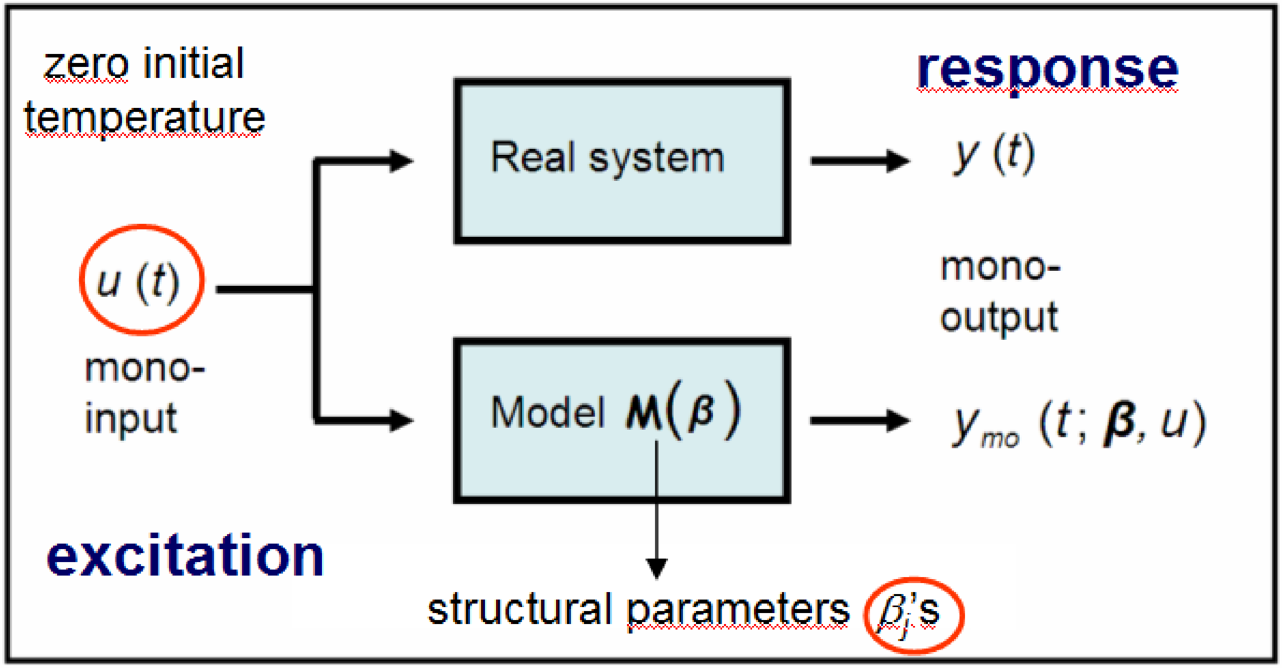
3. State-space representation, model terminology & structure, measurements

4. Different types of inverse problems, measurements & noise, bias

5. Physical model reduction

1. Objectives of a **model** $\mathcal{M}$ (in heat transfer)

First objective: Simulation of physical reality = **Direct Problem**

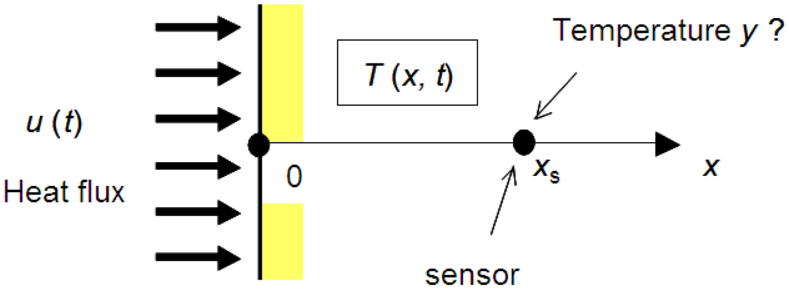


$u(t)$: heat source/flux, variable external temperature

y : measured temperature at given time t and given location

y_{mo} : modelled temperature at given time t and given location

Example: model of a semi infinite medium (1 input - 1 output)



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t}$$

$$-k\,\frac{\partial T}{\partial x}(x, t = 0) = u(t)$$

$$T \text{ finite as } x \to \infty$$

$$T(x, t = 0) = T_0(x)$$

state equations

Perfect temperature sensor: $y = y^{exact}$

Output equation: $y_{mo}(t) = T(x_s, t)$

Solution of direct problem: $y_{mo}(t) = y_{mo\ \text{relax}}(t) + y_{mo\ \text{forced}}(t)$

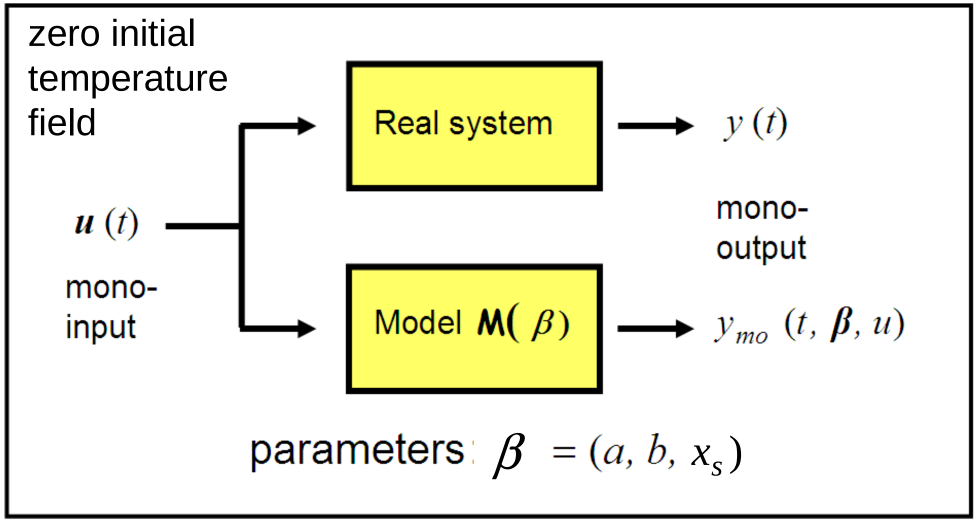
External representation: $y_{mo}(t) = \int_0^{\infty} G(x_s, x, t)\, T_0(x)\, \mathrm{d}x + \int_0^{t} Z(t-\tau)\, u(\tau)\, \mathrm{d}\tau$

Green's function **convolution product**

$$G(x_s, x, t) = \frac{1}{2\sqrt{\pi\, a\, t}}\left[\exp\left(-\frac{(x_s - x)^2}{4\,a\,t}\right) + \exp\left(-\frac{(x_s + x)^2}{4\,a\,t}\right)\right] \quad Z(t) = \frac{1}{b\sqrt{\pi\, t}}\exp\left(-x_s^2/4\,a\,t\right)$$

$$a = k/\rho c$$

$$b = \sqrt{k\,\rho\,c}$$



Thermal impedance:

$$Z(t) = \frac{1}{b\sqrt{\pi\, t}} \exp\left(-x_s^2 / 4\,a\,t\right)$$

$$\bar{Z}\ (p) = \frac{1}{b\sqrt{p}} \exp\left(- x_s \sqrt{p/a}\right)$$

p = Laplace parameter (s^{-1})

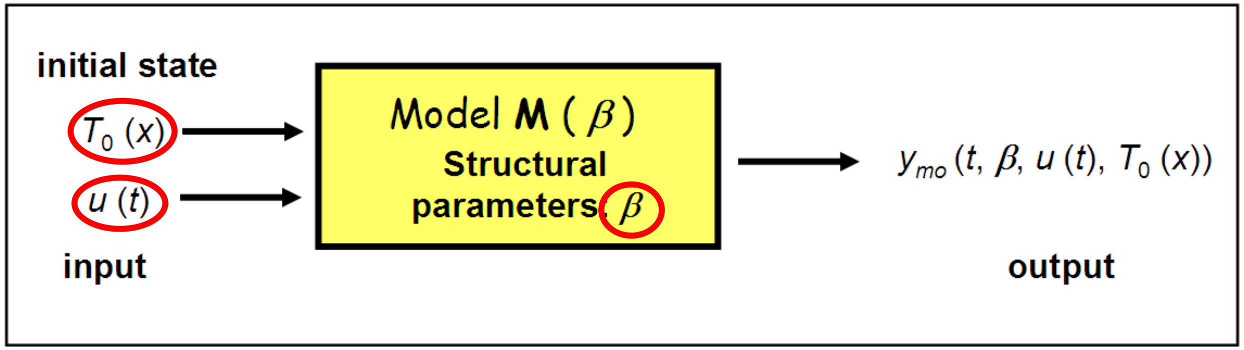
$$\bar{y}_{mo\ \text{forced}}(p) = \bar{Z}(p)\ \bar{u}(p) \ \text{ with } \ \bar{f}\ (p) = \int_0^{\infty} f(t) \exp(-p\,t)\ \mathrm{d}t$$

Parameter list $\beta = (a, b, x_s)$ ➡ Parameter « vector » $\boldsymbol{\beta} = \begin{bmatrix} a \\ b \\ x_s \end{bmatrix}$

a in W m^{-2} - b in J m^{-2} K^{-1} $s^{-1/2}$ - x_s in m $\Rightarrow \|\boldsymbol{\beta}\| = \left(a^2 + b^2 + x_s^2\right)^{1/2}$

M of the white box type (internal representation) :
internal parameters of physical nature

Semi-infinite medium model: general case



List of data of Direct Problem: $X = \{\ \boldsymbol{\beta}\ ,\ u(.)\ ,\ T_0(.)\ \}$

$$\boldsymbol{\beta} = \begin{bmatrix} a \\ b \\ x_s \end{bmatrix}$$

- $\boldsymbol{\beta}$: structural parameter **vector**
- $u(.)$: input (stimulus)
- $T_0(.)$: initial state

input (stimulus) and initial state: **functions**

2. Problem of function parameterizing & Parcimony principle

$$u(t) = \sum_{j=1}^{\infty} u_j f_j(t) \qquad \text{with } \{f_1\ f_2\ \cdots\}$$

function on $[\,0\ \ t_{sup}\,]$

basis of infinite number of functions: $[\,0\ \ t_{sup}\,] \to \boldsymbol{R}$

$$\boldsymbol{u} = [u_1\ \ u_2\ \ \cdots]^T \qquad \boldsymbol{f}(t) = [f_1(t)\, f_2(t) \cdots]^T \qquad \Rightarrow \quad u(t) = \boldsymbol{u}^T \boldsymbol{f}(t)$$

column vectors with an infinite number of components

$$u(t) = \sum_{j=1}^{\infty} u_j f_j(t) \quad \Rightarrow \quad u_j = \langle u(t),\ f_j(t) \rangle$$

projection of $u(t)$ onto $f_j(t)$

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in practice:

$$u_{\text{param}}(t) = \sum_{j=1}^{n} u_j f_j(t) \neq u(t) \quad \Rightarrow \quad u(t) \ \text{ replaced by } \ \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

truncation (n)

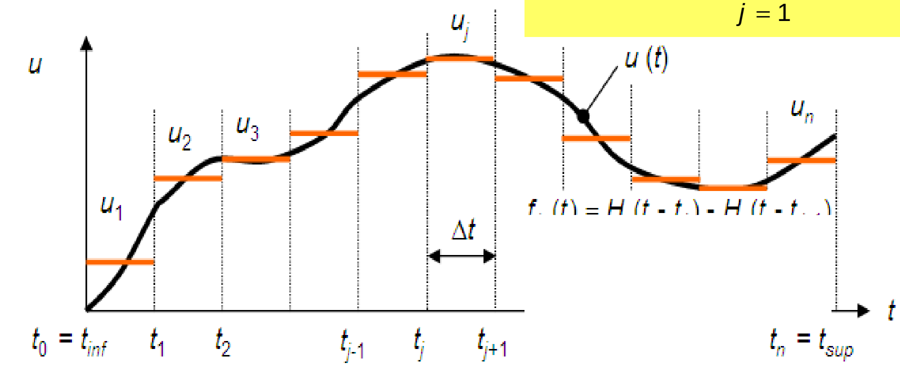
Good approximation:
high $n \Rightarrow$ large number of parameters

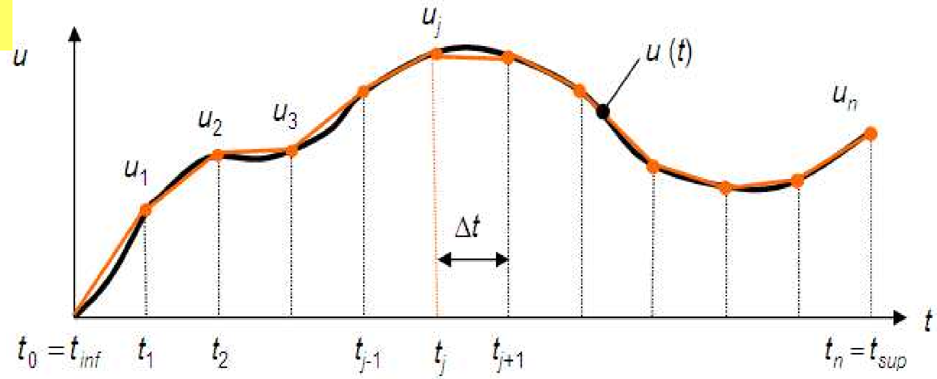
The model-builder has to choose 2 things:

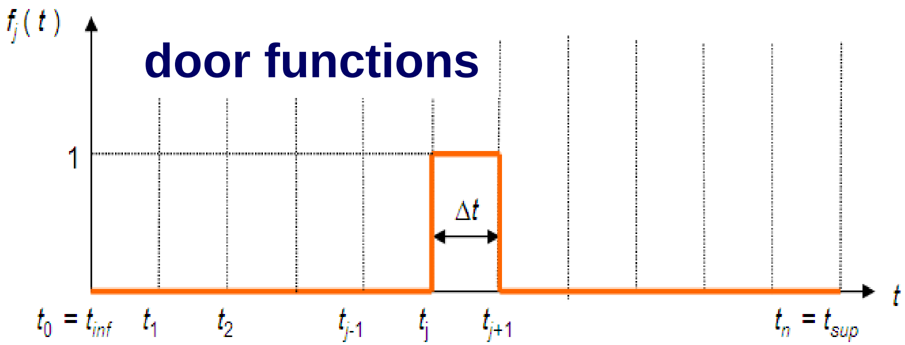
1) functions f_j
2) their number n

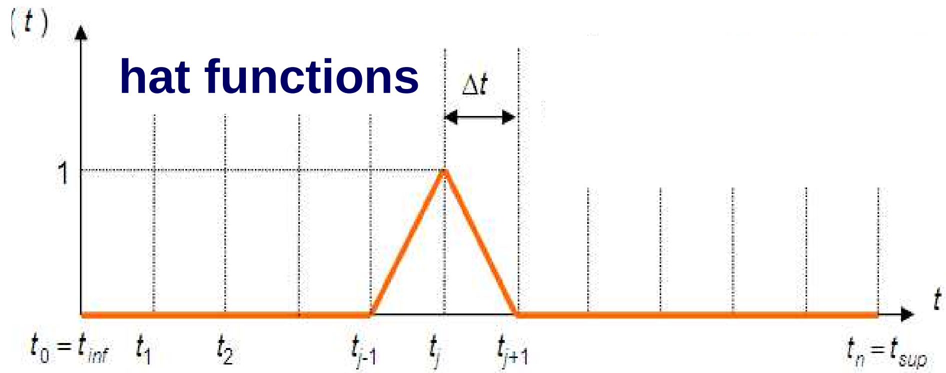
Parameterization: 2 possible choices of *local* function basis

$$u_{\text{param}}(t) = \sum_{j=1}^{n} u_j f_j(t)$$









u_j : averaged value over an interval

• interesting for :

$u\ (t)$ → **linear excitation**

u_j : local discretized value → interpolated $u_{\text{param}}(t)$

• interesting for :

$u\ (T)$ → **non linear excitation**

$\beta(T)$ → **temperature dependent thermophysical properties**

$\beta(\text{P})$ → **heterogeneous material**

$T_0(\text{P})$ → **non uniform initial temperature**

Parameterization (continued)

Remarks

- *non-local* bases available: eigenfunctions f_j of the heat equation (method of separation of variables)

$$u_{\text{param}}(t) = \text{ Fourier series, for example}$$

- orthogonal bases (with a unit function norm N_j) interesting: $\int_{t_{inf}}^{t_{sup}} f_j(t)\, f_k(t)\, dt = N_j\, \delta_{jk}$

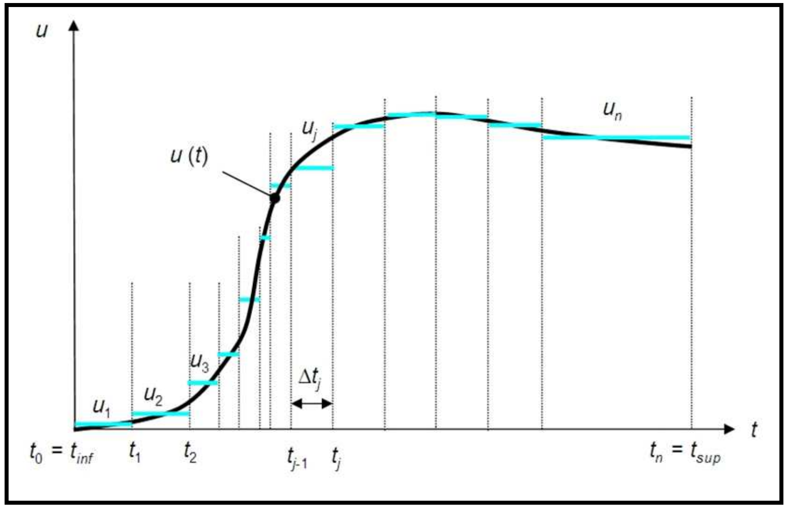
- non constant time step possible

- Extended parameter vector $\boldsymbol{x}$ gathering the Direct Problem data:

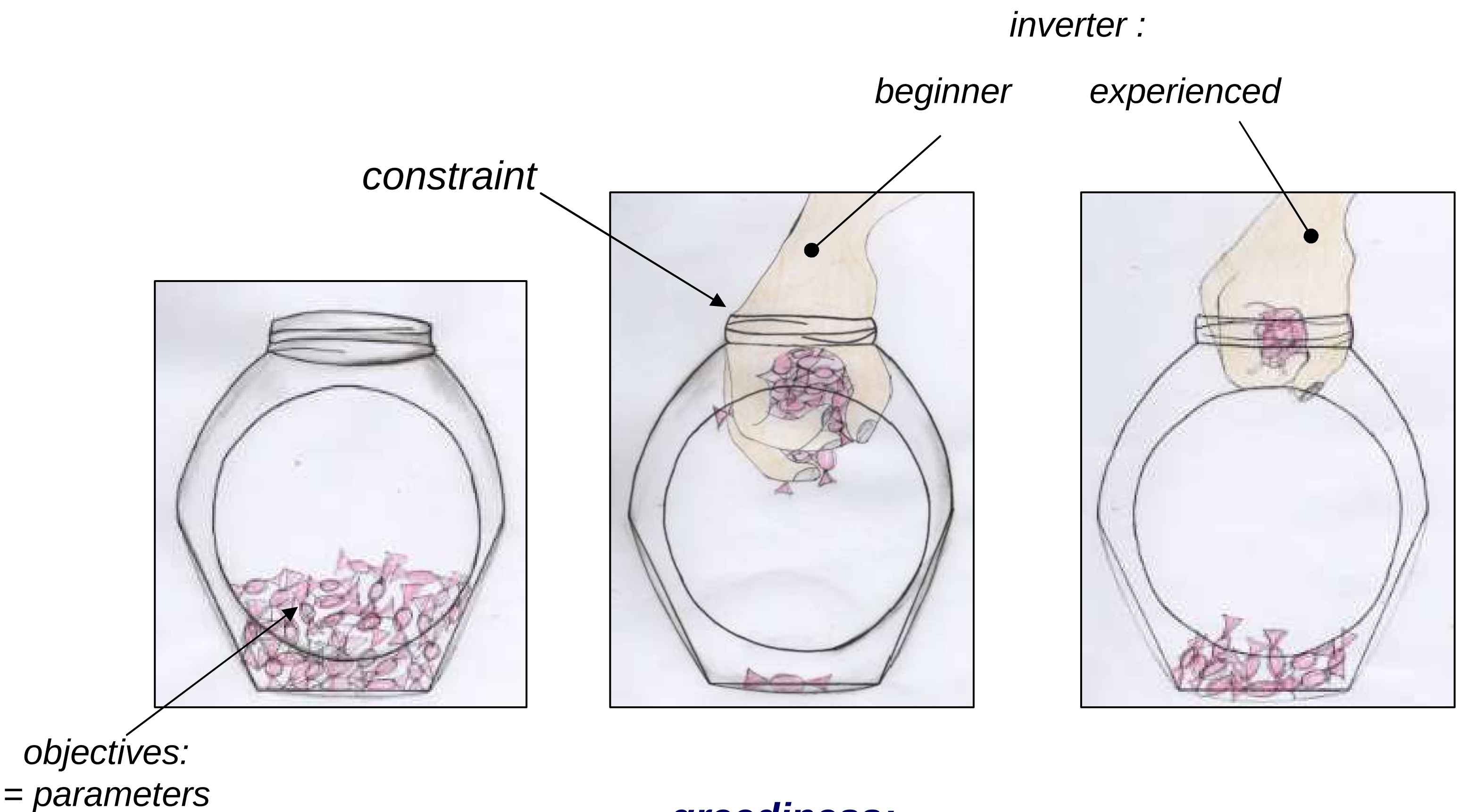
$$x = \{\, \boldsymbol{\beta}\,,\ u(.)\,,\ T_0(\mathrm{P})\,\} \ \rightarrow\ \boldsymbol{x} = \begin{bmatrix} \boldsymbol{\beta} \\ \boldsymbol{u} \\ \boldsymbol{T}_0 \end{bmatrix}$$

list

parameterized functions



Parcimony principle: limitation of the number of parameters
(or of degrees of freedom) to be sought



ideal parameterization

greediness: too many parameters sought / quality of measurements or available a priori information

realistic parameterization

3. State-space representation, model terminology & structure, measurements

$$\mathrm{div}\left(\overline{\overline{\boldsymbol{k}}}\ \mathbf{grad}\ T\right) + q_{vol} = \rho c \frac{\partial T}{\partial t} \qquad + \text{Boundary, interface and initial conditions}$$

W/m³

distributed parameter system

State of the system = continuous temperature field: $T\ (\mathrm{P}, t) = T_{\mathrm{P}}\ (t)$

Discretized state becoming = vector : $\boldsymbol{T}(t) = \left[T_1(t) \quad T_2(t) \quad \cdots \quad T_N(t)\right]$

in a N dimension space (number of nodes)

lumped parameter system:

$$\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}t} = \boldsymbol{E}\ (t, \boldsymbol{T}, \boldsymbol{U}) \qquad \text{with} \quad \boldsymbol{T}\ (t = t_0) = \boldsymbol{T}_0$$

Linear heat source (excitation): $q_{vol}\ (\mathrm{P}, t) \longrightarrow \quad \boldsymbol{U}(t) = \left[u_1(t), u_2(t) \cdots u_p(t)\right]^T$

***p* excitated nodes**

Non-linear heat source: $q_{vol}\ (T\ (\mathrm{P}, t)) \longrightarrow \quad \boldsymbol{U}(t) = \left[u(T_1(t)), u(T_2(t)) \cdots u(T_p(t))\right]^T$

***p* excitated nodes**

State-space representation (continued)

Case of a linear heat source with temperature independent thermophysical properties and coefficients

$$\boldsymbol{E}\,(t, \boldsymbol{T}, \boldsymbol{U}) = \boldsymbol{A}\,\boldsymbol{T} + \boldsymbol{B}\,\boldsymbol{U} \quad \text{with } \boldsymbol{A} \text{ and } \boldsymbol{B}\text{: constant matrices}$$

State equation

$$\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}t} = \boldsymbol{A}\,\boldsymbol{T} + \boldsymbol{B}\,\boldsymbol{U} \qquad \boldsymbol{T}\,(t = t_0) = \boldsymbol{T}_0$$

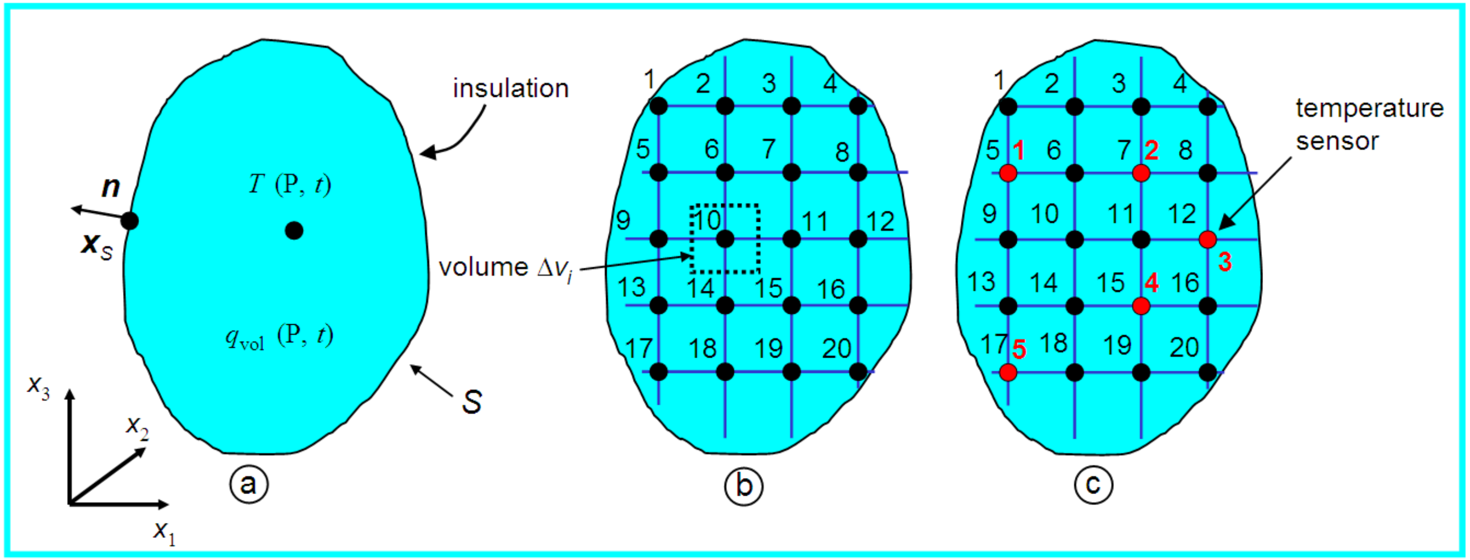
- $\boldsymbol{A}$: state or evolution matrix (N x N)
- $\boldsymbol{T}$: state vector (N x 1)
- $\boldsymbol{B}$: input matrix (N x p)
- $\boldsymbol{U}$: input or control vector (p x 1)
- $\boldsymbol{T}_0$: initial state (N x 1)

Output equation

$$\boldsymbol{y}_{mo} = \boldsymbol{C}\,\boldsymbol{T}$$

- $\boldsymbol{y}_{mo}$: output vector (q x 1)
- $\boldsymbol{C}$: output matrix (q x N) (selection of **observed** nodes)
- $\boldsymbol{T}$: state vector (N x 1)

Output equation: detailed



$$\mathbf{y}_{mo}(t) = \begin{bmatrix} y_{mo1}(t) \\ y_{mo2}(t) \\ y_{mo3}(t) \\ y_{mo4}(t) \\ y_{mo5}(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1(t) \\ T_2(t) \\ T_3(t) \\ T_4(t) \\ T_5(t) \\ \vdots \\ T_{17}(t) \\ T_{18}(t) \\ T_{19}(t) \\ T_{20}(t) \end{bmatrix}$$

q = 5 observed temperatures (output)

N = 20 nodes

Linear state equation : $\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}t} = \boldsymbol{A}\,\boldsymbol{T} + \boldsymbol{B}\,\boldsymbol{U}$

Explicit solution for temperature field:

$$\boldsymbol{T}(t) = \exp\left(\boldsymbol{A}\,(t - t_0)\right)\boldsymbol{T}_0 + \int_{t_0}^{t} \exp\left(\boldsymbol{A}\,(t-\tau)\right)\boldsymbol{B}\,\boldsymbol{U}(\tau)\,\mathrm{d}\tau$$

and for model output:

$$\boldsymbol{y}_{mo}(t) = \boldsymbol{C}\exp\left(\boldsymbol{A}\,(t - t_0)\right)\boldsymbol{T}_0 + \boldsymbol{C}\int_{t_0}^{t} \exp\left(\boldsymbol{A}\,(t-\tau)\right)\boldsymbol{B}\,\boldsymbol{U}(\tau)\,\mathrm{d}\tau$$

Relaxation of initial state **forced (convolution) response**

Remarks:

- advection case possible (dispersion in porous medium, one-temperature model):

$$\mathrm{div}\left(\overline{\overline{\lambda}}\ \mathbf{grad}\ T\right) - \boxed{\rho\, c_f\ \mathbf{v}\ .\ \mathbf{grad}\ T} + q_{\mathrm{vol}} = \rho\, c\,\frac{\partial T}{\partial t} \quad + \text{Boundary,interface and initial conditions}$$

- coupled modes transfer : radiation in semitransparent absorbing medium (Heat equation + radiative transfer equation) ⇒ composite state $\boldsymbol{X}$:

$$\boldsymbol{X}(t) = \begin{bmatrix} \boldsymbol{T}(t) \\ \boldsymbol{I}(t) \end{bmatrix}$$

$\boldsymbol{T}(t)$ ⟶ **Discretized temperature field: position**

$\boldsymbol{I}(t)$ ⟶ **Discretized intensity field: wavelength, direction, position**

- steady state case (linear) : $\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}t} = \boldsymbol{A}\,\boldsymbol{T} + \boldsymbol{B}\,\boldsymbol{U} = \mathbf{0}$ $\quad \boldsymbol{T} = -\boldsymbol{A}^{-1}\boldsymbol{B}\,\boldsymbol{U} \;\Rightarrow\; \boldsymbol{y}_{mo} = -\boldsymbol{C}\,\boldsymbol{A}^{-1}\boldsymbol{B}\,\boldsymbol{U}$

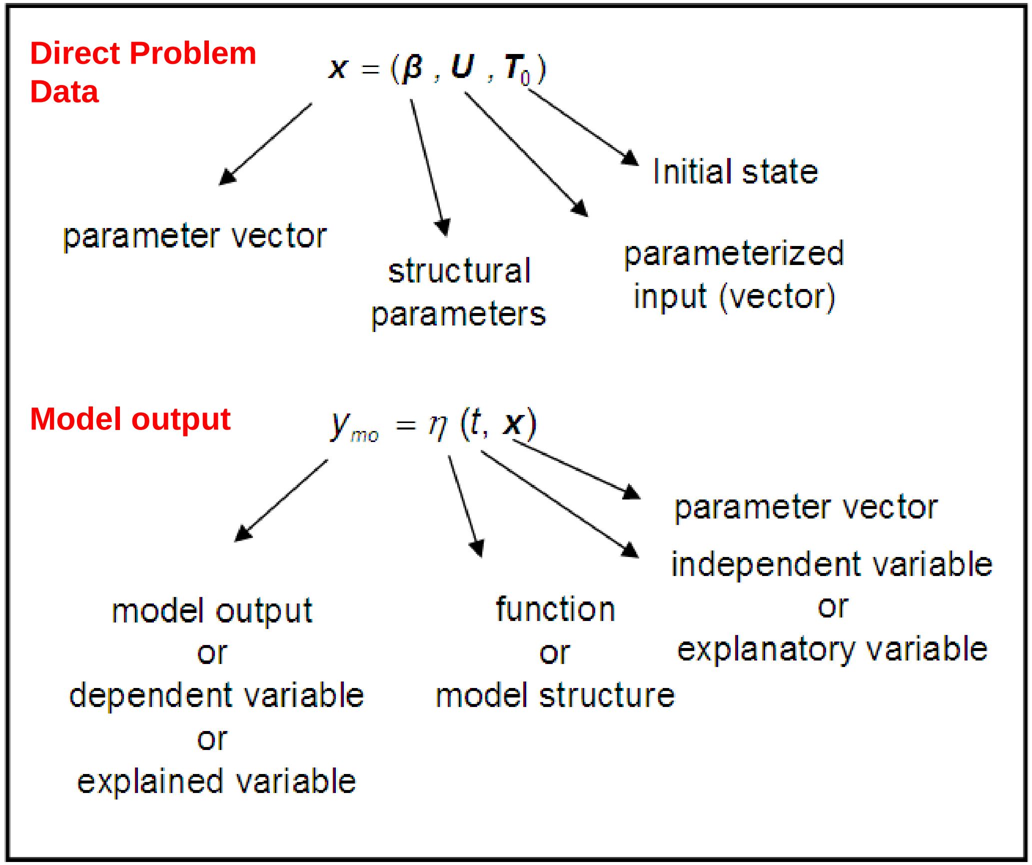
Model terminology and structure

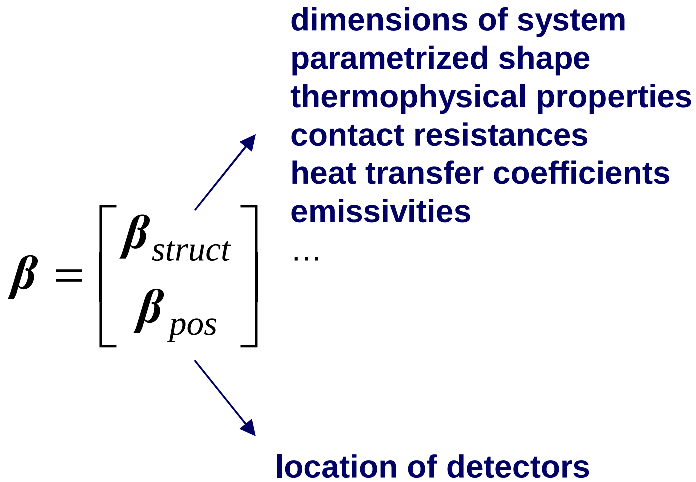
	Before parameterization	After parameterization
• single output:	$y_{mo} = \eta(t, x)$ y_{mo}: scalar; x: list	$y_{mo} = \eta(t, \boldsymbol{x})$ y_{mo}: scalar; $\boldsymbol{x}$: extended parameter vector
• multiple output:	$\boldsymbol{y}_{mo} = \boldsymbol{\eta}(t, x)$ $\boldsymbol{y}_{mo}$: column vector; x: list	$\boldsymbol{y}_{mo} = \boldsymbol{\eta}(t, \boldsymbol{x})$ $\boldsymbol{y}_{mo}$: column vector; $\boldsymbol{x}$: extended parameter vector
• data	$x = \{\boldsymbol{\beta}\,,\; u(.)\,,\; T_0\,(\mathrm{P})\}$	$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{\beta} \\ \boldsymbol{u} \\ \boldsymbol{T}_0 \end{bmatrix}$

$\eta\,(t, \mathbf{.})$ or $\boldsymbol{\eta}\,(t, \mathbf{.})$: scalar or vector function
x or $\boldsymbol{x}$: corresponding data (list/vector)

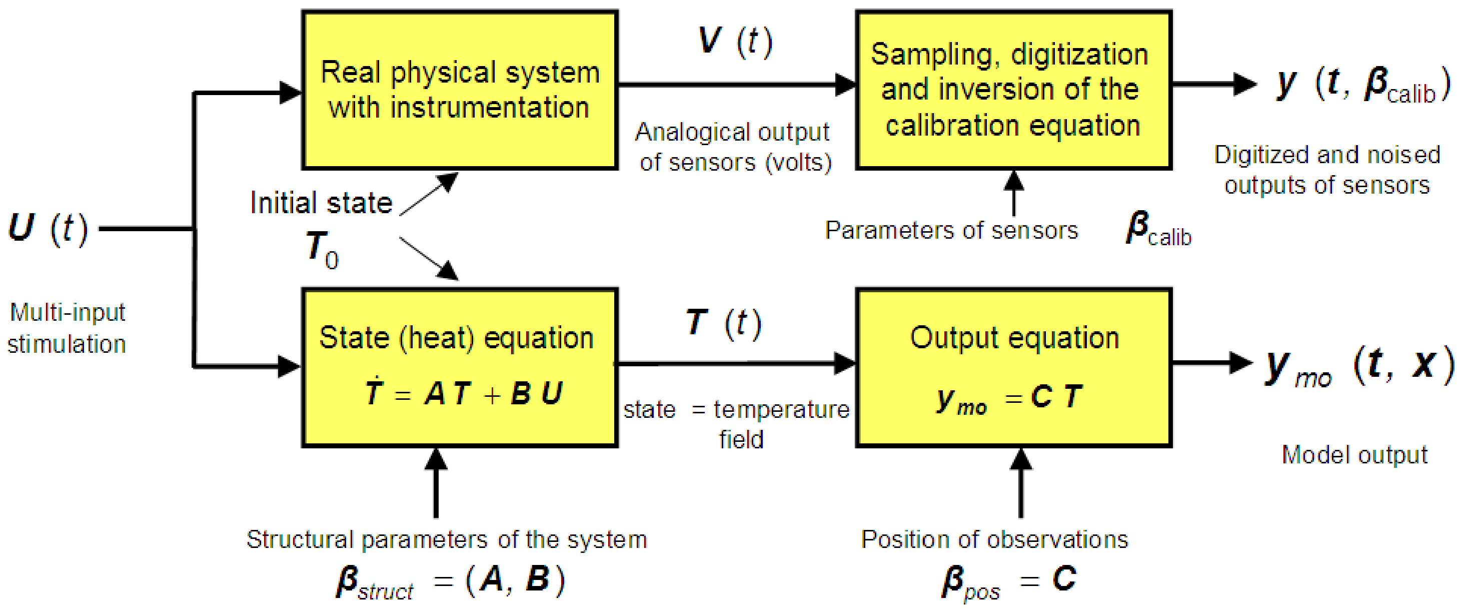
= *structure* of the model

Structure of a parameterized model in heat transfer





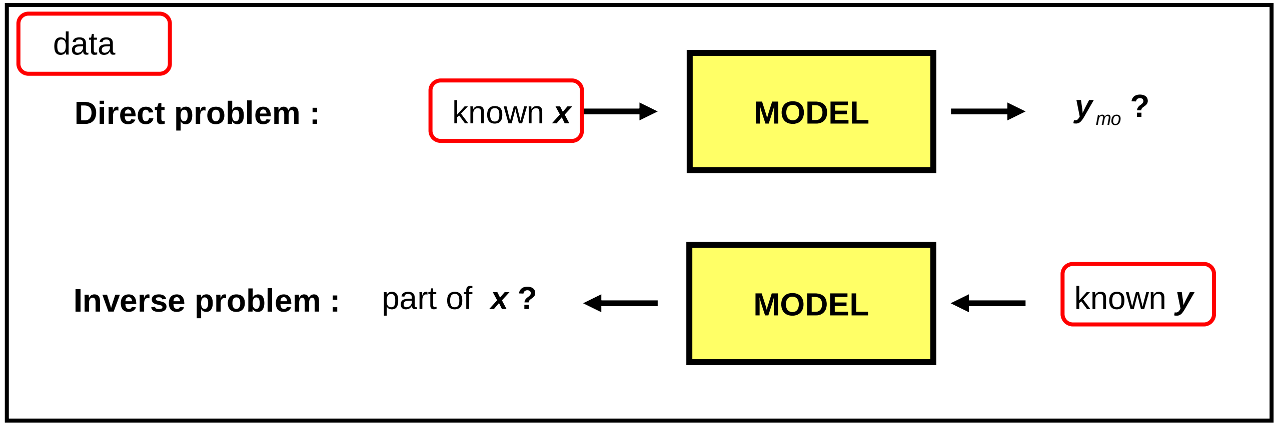
Comparison between measurements and state model



Data of direct problem: $\boldsymbol{x} = \left(\boldsymbol{\beta}_{struct}, \boldsymbol{\beta}_{pos}, \boldsymbol{U}(t), \boldsymbol{T}_0\right)$

Vector of measuring times : $\boldsymbol{t} = \left[t_1 \;\; t_2 \;\; \cdots \;\; t_{final}\right]^T$

Direct and inverse problems



Objective of inverse problem: finding a part $\boldsymbol{x}_r$ of $\boldsymbol{x}$, using additional information (output y or something else)

Extended parameter vector: $$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{\beta}_{struct} \\ \boldsymbol{\beta}_{pos} \\ \boldsymbol{U}(t) \\ \boldsymbol{T}_0 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_r \\ \boldsymbol{x}_c \end{bmatrix}$$

- $\boldsymbol{x}_r$ → sought (**r**esearched) parameters
- $\boldsymbol{x}_c$ → **c**omplementary part: known

4. Different types of inverse problems in heat transfer, measurements & noise, bias

- ***inverse measurement problems*** → additional information stems from output signal $\boldsymbol{y}$ of sensors

- ***control problems***

  → additional information = desired (target) values of state $\boldsymbol{T}$ or output $\boldsymbol{y}$

  → sought quantity = excitation $\boldsymbol{U}$, initial state $\boldsymbol{T}_0$ or velocity/flowrate in $\boldsymbol{\beta}$

- ***system identification problems*** = model construction

  - ***model reduction***

  → additional linformation = output of detailed model $\eta_{\text{det}}\left(t\,;\,\boldsymbol{x}_{\text{det}}\right)$

  → sought quantity = structure + parameter vector of a reduced model $\eta_{\text{det}}\left(t\,;\,\boldsymbol{x}_{\text{det}}\right) \approx \eta_{\text{red}}\left(t\,;\,\boldsymbol{x}_{\text{red}}\right)$

$$\text{with}:\ \boldsymbol{x}_{\text{det}} = \begin{bmatrix}\boldsymbol{\beta}_{\text{det}} & \boldsymbol{U}_{\text{det}} & \boldsymbol{T}_{0\,\text{det}}\end{bmatrix}^T \ \text{ and }\ \boldsymbol{x}_{\text{red}} = \begin{bmatrix}\boldsymbol{\beta}_{\text{red}} & \boldsymbol{U}_{\text{red}} & \boldsymbol{T}_{0\,red}\end{bmatrix}^T$$

1) *mathematical reduction* :

GREY BOX type

$$u_{\text{red}}(\text{P},t) = u_{\text{det}}(\text{P},t) \quad \Rightarrow \quad \boldsymbol{U}_{\text{red}} = \boldsymbol{U}_{\text{det}}$$
$$T_{0\,\text{red}}(\text{P},t) = T_{0\,\text{det}}(\text{P},t) \quad \Rightarrow \quad \boldsymbol{T}_{0\,\text{det}} = \boldsymbol{T}_{0\,\text{red}}$$

2) *physical reduction* :

WHITE BOX type

$$u_{\text{red}}(\text{P},t) \approx u_{\text{det}}(\text{P},t) \quad \Rightarrow \quad \boldsymbol{U}_{\text{red}} = f_U\left(\boldsymbol{U}_{\text{det}}\right)$$
$$T_{0\,\text{red}}(\text{P},t) \approx T_{0\,\text{det}}(\text{P},t) \quad \Rightarrow \quad \boldsymbol{T}_{0\,\text{red}} = f_{T0}\left(\boldsymbol{T}_{0\,\text{det}}\right)$$

In both cases: $\boldsymbol{\beta}_{\text{red}} = f_\beta\left(\boldsymbol{\beta}_{\text{det}}\right)$ ⟶ *explicit for physical reduction*

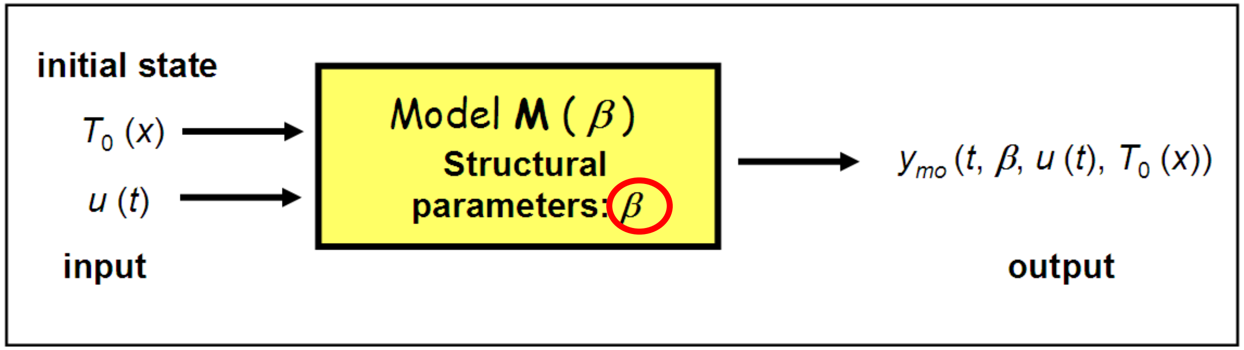
-***experimental model identification*** (belongs to inverse measurements problem class)

→ additional information = ***y***, ***U*** and T_0 are measured, or supposed to be known

→ sought quantity = parameter vector $\boldsymbol{\beta}$ for a model of given structure

3 types of identified models:

- *white box* type, based on first principles: physical meaning for $\boldsymbol{\beta}$
- black box type: general structure, no physical meaning for $\boldsymbol{\beta}$ (neural networks)
- *grey box* type : in between, physical structure, no physical meaning for $\boldsymbol{\beta}$



- ***optimal design problems***

→ additional information: quality criterion to satisfy

→ sought quantity = parameter vector $\boldsymbol{\beta}$ with constaints, for a model of given structure

Inverse measurement problems in heat transfer

Extended parameter vector: $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{\beta}_{struct} \\ \boldsymbol{\beta}_{pos} \\ \boldsymbol{U}(t) \\ \boldsymbol{T}_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_r \\ \boldsymbol{x}_c \end{bmatrix}$ → sought (researched) parameters / complementary part: known

Measurements $\boldsymbol{y}(t)$ available on $[t_0, t_{final}]$ interval

a) Inverse problems of ***structural parameters estimation*** : $\boldsymbol{x}_r \equiv \boldsymbol{\beta}_r$

- example 1: thermophyscal property « measurement »: $x_r = k$ or ρc or $a ... + h$!

- example 2: calibration of a sensor/acquisition chain: $V_{mo}(T, \boldsymbol{\beta}_{calib})$

b) Inverse ***input problems*** : $x_r \equiv u(\mathrm{P}, t)$

- example: “inverse heat conduction” = wall heat flux “measurement”

c) Inverse ***initial state problems*** : $x_r \equiv T_0(\mathrm{P})$

d) Inverse ***shape reconstruction*** problems

e) Inverse problems of ***optimal design/control***

(of a characterzation experiment, for example)

Measurement and noise

$$y(t) = y^{exact}(t) + \varepsilon(t)$$

measured temperature signal (sensor output)

« true » temperature (unknown)

Temperature measurement noise (unknown)

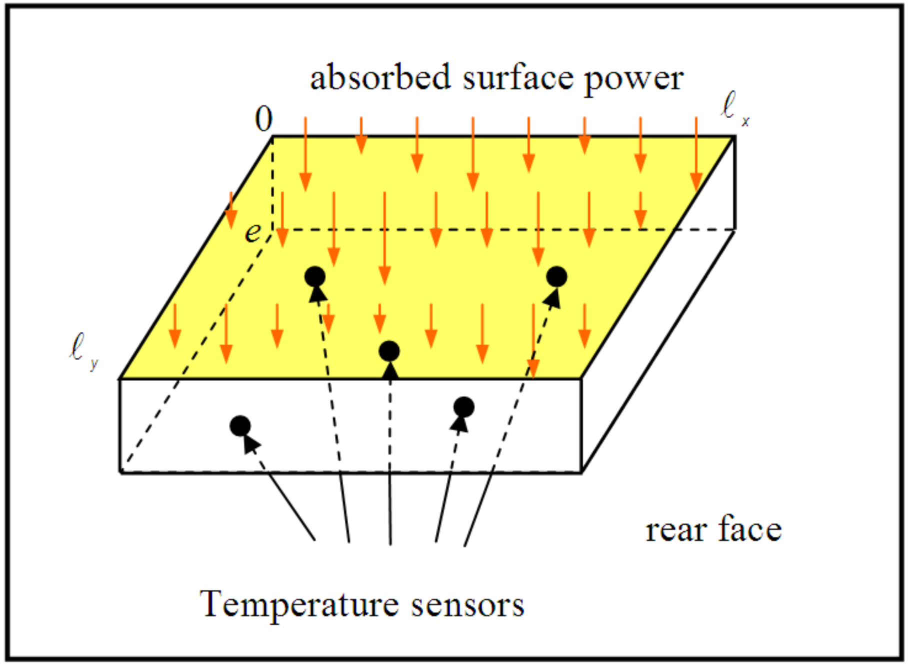
- only discrete values available $y_i = y(t_i) \quad \Rightarrow \varepsilon_i = \varepsilon(t_i)$

- **(implicit) assumptions:**

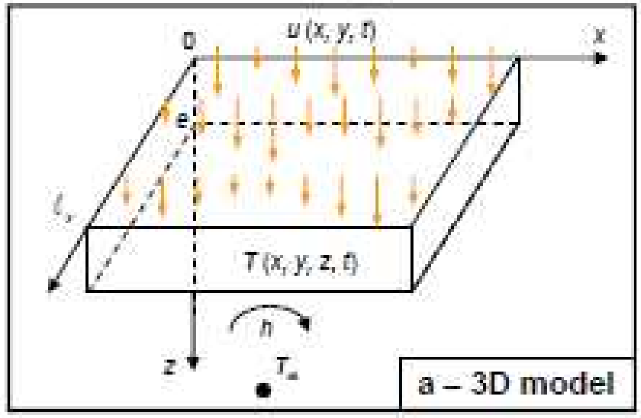
→ unbiased model: $y^{exact}(t) = y_{mo}(t, \boldsymbol{x}^{exact})$

→ unbiased noised: $E(\varepsilon_i) = 0$

5. Physical model reduction on an example



- homogeneous rectangular slab, thickness *e*, lengths
- thermal diffusivity and conductivity *a* and *k*, volumetric heat $\rho c = k / a$
- 4 lateral sides insulated, *h* heat exchange coefficient over rear face
- uniform initial temperature T_0
- two dimensional heat flux absorption over front face
- *q* temperature sensors inside the slab



a – 3D model

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{a}\frac{\partial T}{\partial t}$$

input quantities

$$T = T_0 \quad \text{for} \quad t = 0$$

$$\frac{\partial T}{\partial x} = 0 \quad \text{at} \quad x = 0, \ell_x \; ; \; \frac{\partial T}{\partial y} = 0 \quad \text{at} \quad y = 0, \ell_y$$

$$-k\frac{\partial T}{\partial z} = u(x, y, t) \quad \text{at} \quad z = 0 \; ; \; -k\frac{\partial T}{\partial z} = h\,(T - T_\infty) \quad \text{at} \quad z = e$$

explanatory variable

$$y_{mo,i} = \eta_i\,(t\; x) = T_a\,(x_i, y_i, z_i, t \,;\, u\,(x, y, t); T_0, T_\infty, h, \ell_x, \ell_y, e\; k\,, a)$$

output = observation at sensor i location (model **Ma**)

β

(3 q + 9) quantities

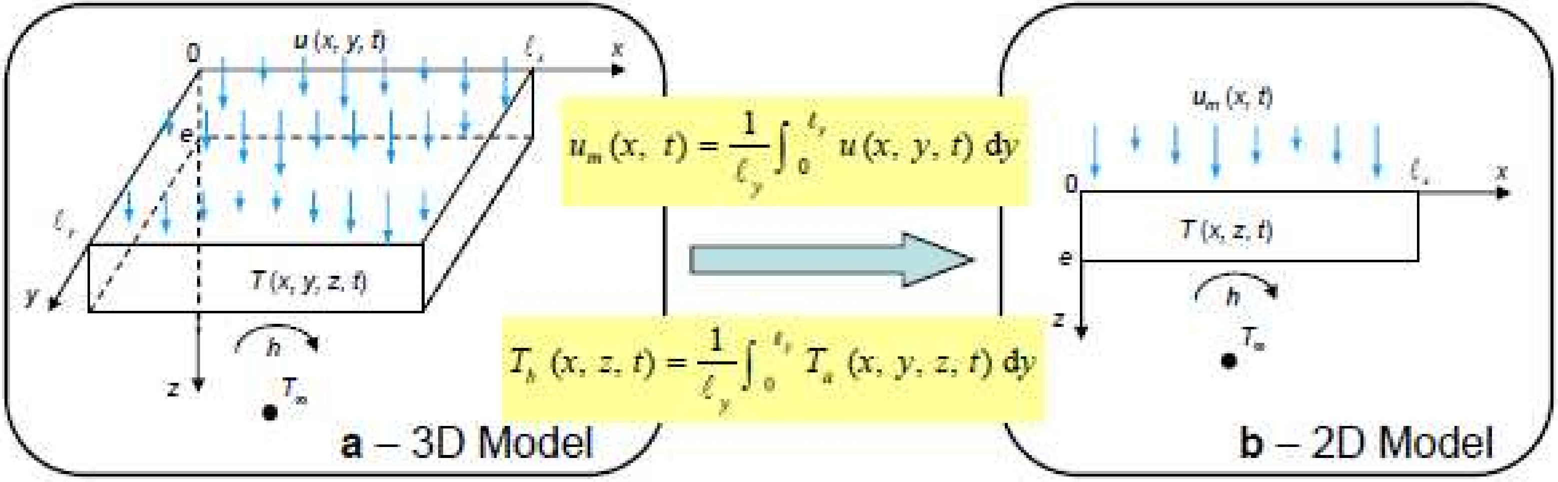
→ **Dimensionless form:** $T^* = (T - T_\infty)/\Delta T$

$$y_{mo,i} = \eta_*(t, x_*) = \Delta T \,.\, T^*(x_i^*, y_i^*, z_i^*, t/\tau_{\text{diff}}, R, u\,(x, y, t)/\Delta T, H, \ell_x^*, \ell_y^*) + T_\infty$$

$\Delta T = T_0 - T_\infty \quad x_i^* = x_i/e \quad y_i^* = y_i/e \quad z_i^* = z_i/e \quad \ell_x^* = \ell_x/e \quad \ell_y^* = \ell_y/e \quad R = e/k \quad \tau_{diff} = e^2/a \quad H = he/k$

parameter list: $x^* = (\beta^*, u, \Delta T, T_\infty)$ (3 q + 8) quantities or (+9)

structura/positions parameter vector: $\beta^* = ((x_i^*, y_i^*, z_i^*), \text{ for } i = 1 \text{ to } q), \; \tau_{diff}, \; R, H, \ell_x^*, \ell_y^*)$



$$u_m(x,\ t) = \frac{1}{\ell_y}\int_0^{\ell_y} u(x,\ y,\ t)\ dy$$

$$T_b(x,\ z,\ t) = \frac{1}{\ell_y}\int_0^{\ell_y} T_a(x,\ y,\ z,\ t)\ dy$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{a}\frac{\partial T}{\partial t} \qquad T = T_0 \quad \text{at} \quad t = 0$$

$$-k\frac{\partial T}{\partial z} = u_m(x,\ t) \quad \text{at} \quad z = 0\ ; \quad -k\frac{\partial T}{\partial z} = h\ (T - T_\infty) \quad \text{at} \quad z = e$$

Model b

Observations :

$$y_{mod,\,i}\ (t_k) = T_b\ (x_i,\ z_i\ ,\ t_k)$$

$x = \{\boldsymbol{\beta}, u_m, \Delta T, T_\infty\}$ (2 q + 7) quantities

$$\beta = ((x_i^*,\ z_i^*,\ \text{for } i = 1 \text{ à } q),\ \tau_{diff},\ R,\ H,\ \ell_x^*)$$

Measurements

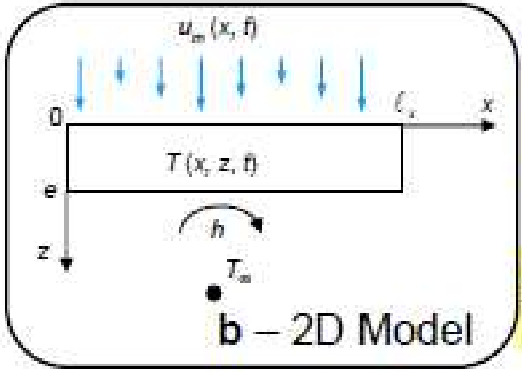
$$y_i\ (t_k) = T_k^{exp}\ (x_i\ ,\ \cancel{y_i},\ z_i)$$

$$y_i\ (t_k) = \frac{1}{n_i}\sum_{j=1}^{n_i} T_k^{exp}\ (x^m,\ y^j = y_i)$$

averaging in y-direction

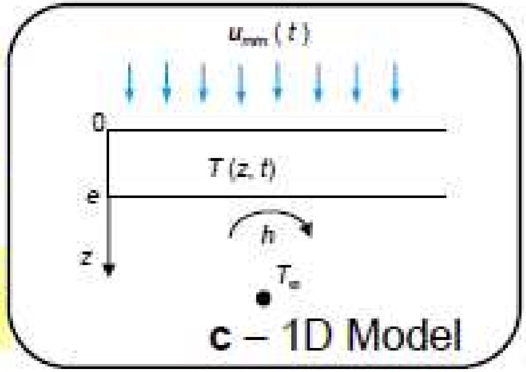
→ $x - t$ temperature signal

y location disappears



$$u_{mm}(t) = \frac{1}{\ell_x}\int_0^{\ell_x} u_m(x,\ t)\,\mathrm{d}x$$

$$T_c\,(z,t) = \frac{1}{\ell_x}\int_0^{\ell_x} T_b\,(x,z,t)\,\mathrm{d}x$$



$$\frac{\partial^2 T}{\partial z^2} = \frac{1}{a}\frac{\partial T}{\partial t}$$

$$\beta = ((x_i^*,\ z_i^*,\ \text{for } i = 1 \text{ à } q),\ \tau_{diff},\ R,\ H,\ \ell_x^*)$$

$$T = T_0 \ \text{ for } \ t = 0$$

$$-k\frac{\partial T}{\partial z} = u_{mm}(t) \ \text{ at } \ z = 0\ ; \ -k\frac{\partial T}{\partial z} = h(T - T_\infty) \ \text{ at } \ z = e$$

Model c

$$y_{mod,i}\,(t_k) = T_c\,(z_i\,,t_k)$$

$$x = \{\boldsymbol{\beta},\ u_{mm},\ \Delta T,\ T_\infty\}$$ (q + 6) quantities

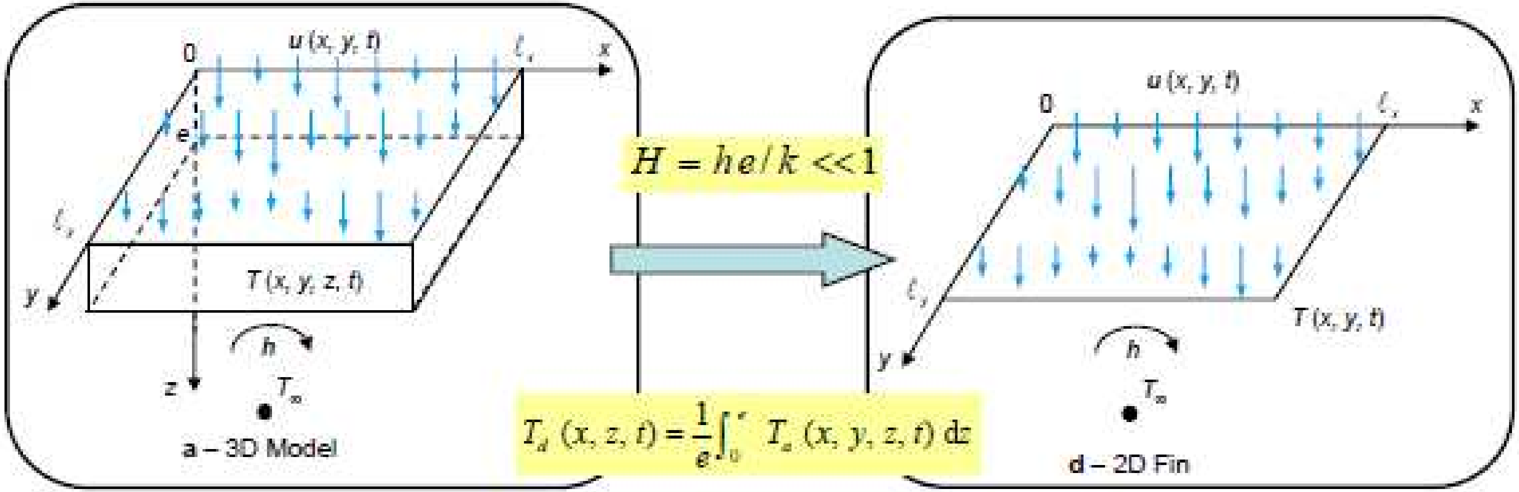
$$\boldsymbol{\beta} = (z_i^*,\ \text{for } i = 1 \text{ à } q),\ \tau_{diff},\ R,\ H)$$

Measurements

$$y\,(t_k) = \frac{1}{q}\sum_{i=1}^{q} T_k^{\exp}(x_i,\ y_i)$$

averaging in *x*- and *y*- direction

→ *t* temperature signal



2D Fin equation: $$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \frac{h(T - T_\infty)}{k\,e} + \frac{u\ (x, y, t)}{k} = \frac{1}{a}\frac{\partial T}{\partial t}$$

$$T = T_0 \quad \text{at} \quad t = 0 \qquad \frac{\partial T}{\partial x} = 0 \quad \text{in} \quad x = 0, \ell_x \ ; \ \frac{\partial T}{\partial y} = 0 \quad \text{in} \quad y = 0, \ell_y$$

Model d

$$y_{mod,i}\ (t_k) = T_d\ (x_i, y_i, t_k)$$

$x = \{\boldsymbol{\beta}, u_m, \Delta T, T_\infty\}$ $(2\,q + 8)$ quantities

$$\boldsymbol{\beta} = ((x_i^*,\ y_i^*,\ \text{for } i = 1 \text{ à } q),\ \tau_{diff},\ R,\ H,\ \ell_x^*,\ \ell_y^*)$$

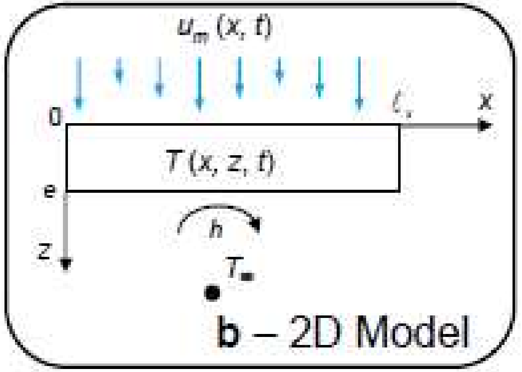
Measurements

$$y_i\ (t_k) = T_k^{exp}\ (x_i, y_i, \not{z_i})$$

or

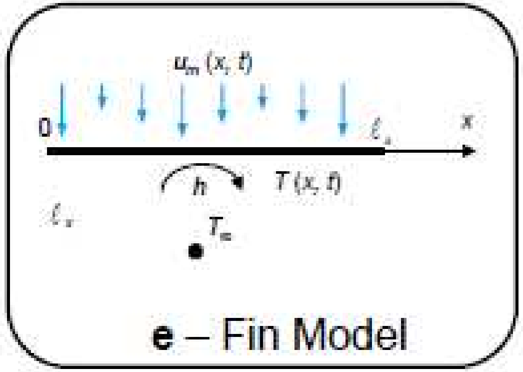
$$y_i\ (t_k) = \frac{1}{n_i}\sum_{p=1}^{n_i} T^{exp}(x_i, y_i, z^p = z_i)$$

z_i disappears



$$T_e(x, t) = \frac{1}{e}\int_0^e T_b(x, z, t)\,dz$$

$$H = he/k \ll 1$$



1D Fin equation:

$$\frac{\partial^2 T}{\partial x^2} - \frac{h(T - T_\infty)}{k\,e} + \frac{u_m(x, t)}{k\,e} = \frac{1}{a}\frac{\partial T}{\partial t}$$

$$T = T_0 \quad \text{at} \quad t = 0 \qquad \frac{\partial T}{\partial x} = 0 \quad \text{in} \quad x = 0, \ell_x$$

Model e

$$y_{mod}(t_k) = T_c(x_i, z_i, t_k)$$

$$x = \{\boldsymbol{\beta}, u_m, \Delta T, T_\infty\}$$ (q + 7) quantities

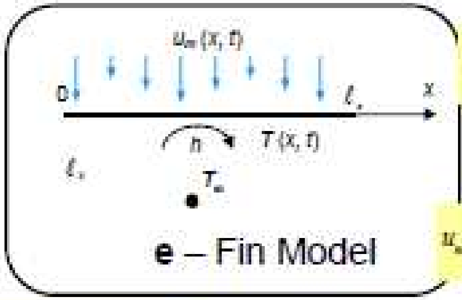
$$\boldsymbol{\beta} = (x_i^*, z_i^*, \text{ for } i = 1 \text{ à } q), \tau_{diff}, R, H)$$

Measurements

$$y_i(t_k) = \frac{1}{n_i}\sum_{j=1}^{n_i} T_k^{exp}(x^m, y^j = y_i)$$

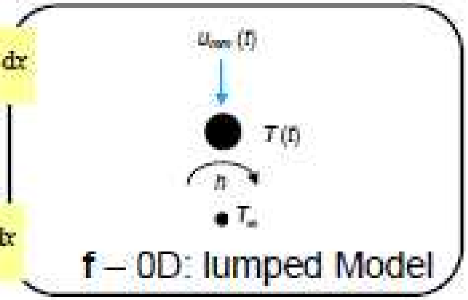
averaging in *y*-direction

$\rightarrow x - t$ temperature signal



$$T_f(t) = \frac{1}{\ell_x}\int_0^{\ell_x} T_s(x,t)\,dx$$

$$u_{mm}(t) = \frac{1}{\ell_x}\int_0^{\ell_x} u_m(x,t)\,dx$$



$$H_x, H_y, H \ll 1$$

$$H_x = h\,\ell_x / k$$

$$H_y = h\,\ell_y / k$$

$$H = h\,e / k$$

Lumped capacitance model:

$$\rho c e \frac{dT}{dt} + h\,(T - T_\infty) = u_{mm}(t)$$

$$T = T_0 \quad \text{at} \quad t = 0$$

$$\tau = \rho c e / h = \tau_{diff} / H \qquad G = 1/h$$

Model f bulk temperature observation

$$T = T_\infty + \Delta T \exp(-t/\tau) + \frac{G}{\tau}\int_0^t u_{mm}(t') \exp\left(-\frac{t-t'}{\tau}\right) dt'$$

$$x = (\beta, u_{mm}, \Delta T, T_\infty) \qquad \Delta T = T_0 - T_\infty$$

$$\beta = (\tau, G)$$

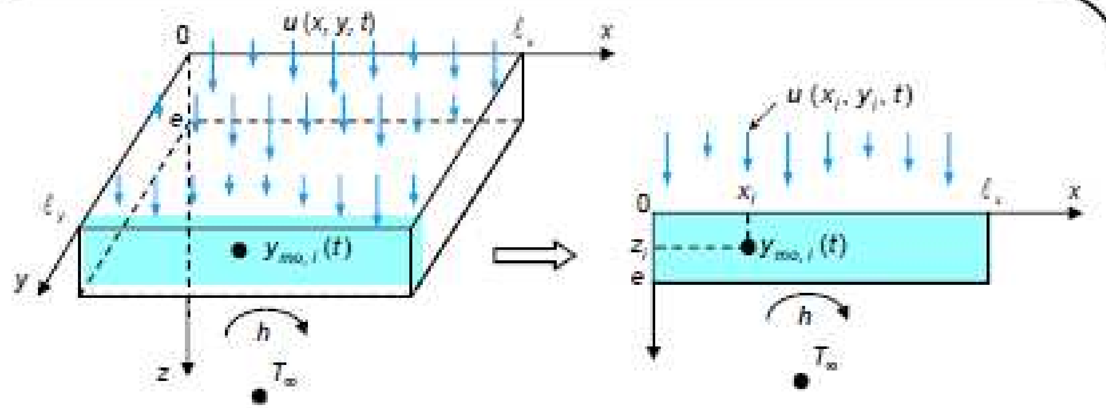
5 independent variables

Measurements

$$y(t_k) = \frac{1}{q}\sum_{i=1}^{q} T_k^{exp}(x_i, y_i)$$

averaging in *x*- and *y*- direction
→ *t* temperature signal

g - 1D *local* Model



$$\frac{\partial T}{\partial x}(x_i, y_i, z_i, t) \approx \frac{\partial T}{\partial y}(x_i, y_i, z_i, t) \approx 0 \quad \Rightarrow \quad y_{mo,i}(t) = T_c\,(z_i, t\,;\, u\,(x_i, y_i, t))$$

Anisotropic material : $k_x = k_y = 0$ $\quad k_z = k$ or $\frac{\partial u}{\partial x} \approx 0$ and $\frac{\partial u}{\partial x} \approx 0$ $\left(\text{weak } \frac{\partial e}{\partial x}, \frac{\partial e}{\partial y}, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \dots\right)$

Model f *q* independent *local* temperature models

$$y_{mo,i} = T_{g,i}\,(t) = T_c(z_i, t\,;\, \beta_i, u(x_i, y_i, t), \Delta T, T_\infty)$$

$$x_i = \{\beta_i,\ u(x_i, y_i, t), \Delta T, T_\infty\}$$

$$\beta_i = (z_i^*, \tau_{diff\,i}, R_i, H_i)$$

(*q* + 6) independent quantities if same z_i's

Measurements

$$y_i\,(t_k) = T_k^{exp}(x_i, y_i, z_i)$$

Remarks on physical reduction (for later inversion)

- the **simpler** the model, the **higher** the possible bias (for direct simulation)

but

- detailed model may be biased too /**experiment**
- decrease in **number** of parameters
- inversion more **robust**/noise amplification (inversion)
- parameters keep **explicit** physical meaning (white box): can be **exported** !
- **first step** for later finer inversion (non linear estimation)

Thank you for your attention !