# Optimizing climate data integration for thermal properties characterization in building walls using thermal quadrupoles

**Mostafa MORTADA[1*], Vincent FEUILLET[1], Laurent IBOS[1], Kamel ZIBOUCHE[2], Julien WAEYTENS[3]**

[1]Univ Paris Est Creteil, CERTES, F-94010 Creteil, France
[2]Centre Scientifique et Technique du Bâtiment (CSTB), 84 Avenue Jean Jaurès, 77420 Champs-sur-Marne, France
[3]Univ Gustave Eiffel, COSYS, IMSE, Cité Descartes, 14-20 Boulevard Newton, 77420 Champs-sur-Marne, France
[*](Corresponding author: mostafa.mortada@u-pec.fr)

**Abstract -** The following work deals with the establishment of a model allowing the incorporation of climate data into the thermal quadrupoles methodology for solving the heat transfer problem in a multilayered building wall. This is performed through fitting these sets of data into suitable functions (Fourier series or discrete Laplace transform) that are manageable in terms of applying the Laplace transform which is required to establish the quadrupole relations. Subsequently, an investigation into the suitable Laplace inversion algorithm is performed alongside parametric analyses to establish the validity and robustness of the model.

## 1. Introduction

Modeling heat diffusion plays a key role in applications ranging from thermodynamic machinery to energy efficiency improvements in buildings. Heat transfer in building walls often relies on numerical tools like COMSOL [1], ANSYS [2], but their high computational cost is still a major drawback. The thermal quadrupole method offers an alternative, where the problem is solved in transformed time and space domains [3]. One advantage of this method is that it facilitates accounting for boundary conditions, also by performing time and space transforms on them. This paper investigated the feasibility of utilizing recorded climate data as a boundary condition to solve the heat equation in multilayered building walls.

The use of real climate data as a boundary condition in the thermal quadrupole model remains a challenge. Existing literature on thermal quadrupole modeling in building field typically assumes constant or predefined functional boundary conditions to simplify the Laplace transforms. Some studies used transfer functions and convolution products to reconstruct solutions [5]. Ginestet et al. [5] applied the quadrupole model in an inverse methodology but encountered difficulties integrating real climate data. Yang et al. [6] obtained transfer functions for the unit-step and unit-pulse responses which are the basis for the evaluation of the walls' performance, while Dlimi et al. [7] and Leccese et al. [8] modeled boundary conditions as sinusoidal functions to facilitate the Laplace transforms.

The proposed study introduces a methodology for integrating real climate data into the thermal quadrupole framework to create a fast and reliable direct model which could be later employed for inverse thermal property estimation. The approach involves fitting climate data (external air temperature and solar flux) into a function suitable for applying Laplace transforms using either a Fourier series or a discrete Laplace transform. The fitted data are incorporated into a 1D quadrupole model, solved in the Laplace domain, and then transformed back to the

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time domain using Gaver-Stehfest, De Hoog, or Den Iseger inversion algorithms. This study is conducted in the framework of the ANR RESBIOBAT project with the objective to improve an existing method for the *in situ* determination of building wall thermal resistance.

The paper is structured as follows: Section 2 presents the thermal quadrupole modeling, data fitting techniques, and Laplace inversion algorithms. Section 3 discusses the results and parametric analysis, while section 4 concludes the study.

## 2. Thermal quadrupole model

The system being analyzed is a multilayered wall, as illustrated in Figure 1, the number $N$ of layers depending on the type of wall being examined. The wall thermal resistance assessment is carried out using an active approach under *in situ* conditions [9]. Therefore, the wall is subjected to heat excitation on its interior surface and weather conditions on its exterior surface.

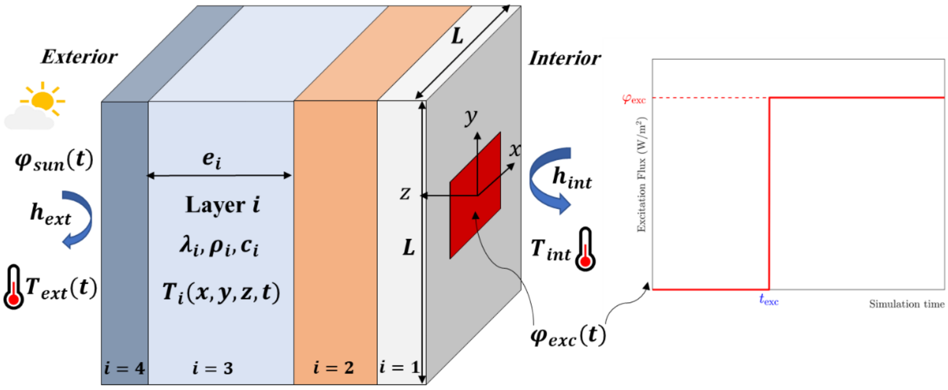



Figure 1: *Multilayered wall model and its interior and exterior boundary conditions*

### 2.1. 1D modeling

The structure in Figure 1 can be simplified to a 1D heat transfer problem under certain assumptions. The excitation heat flux $\varphi_{exc}$ on the interior wall is assumed uniform, with no variations in the $x$ and $y$ directions close to the center of the excitation. The lateral boundaries are considered adiabatic, preventing heat flow in those directions. Boundary conditions are independent of $x$ and $y$ coordinates. Each layer's thermal properties (density $\rho_i$, thermal conductivity $\lambda_i$, and specific heat $c_i$) are considered homogeneous and isotropic, with heat conduction occurring only in the $z$-direction (through the wall thickness). Therefore, the heat equation in cartesian coordinates reduces to:

$$\frac{\partial^2 T_i}{\partial z^2} = \frac{\rho_i c_i}{\lambda_i} \frac{\partial T_i}{\partial t} \text{ for } i = 1 \text{ to } N \tag{1}$$

### 2.2. Climate data incorporation

As previously mentioned, the climate data are incorporated into the quadrupole equations directly through fitting functions that facilitate their Laplace transforms. The data sets correspond to hourly recordings of external air temperature and solar flux for two different French cities. Nancy, located in the northeast of France, is characterized by a Continental climate with colder winters and less solar radiation. In contrast, Carpentras lies in the southeast and experiences a Mediterranean climate, with milder winters and significantly higher solar exposure. These contrasting climatic conditions provide a robust basis for evaluating the proposed method's sensitivity and adaptability.

The climate data were obtained with an hourly acquisition frequency, which is a standard and widely accepted temporal resolution in building physics applications. Nonetheless, the thermal quadrupoles model developed in this study is flexible and can accommodate shorter time steps (i.e., higher frequency data). This adaptability makes the method suitable for high-resolution experimental investigations and dynamic simulations requiring finer temporal detail. Two methods are investigated for their integration into the thermal quadrupoles, a Fourier series fit and a discrete Laplace transform.

### 2.2.1. Fourier series fit

The first suggested fitting technique is a Fourier series, leveraging the data's oscillatory nature, which is the basis for the sinusoidal climate representations encountered in literature. The Fourier series for an $L$-periodic function on $[0, L)$ is given by:

$$f(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos\left(\frac{2\pi k}{L}x\right) + B_k \sin\left(\frac{2\pi k}{L}x\right) \tag{2}$$

$$\text{with } A_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi k}{L}x\right)dx \text{ and } B_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi k}{L}x\right)dx$$

When fitting the climate data, which are hourly recordings of external air temperature $T_{ext}$ and solar flux $P_{solar}$ over a 6-day period for two cities (Carpentras and Nancy), Fourier edge effects are experienced, which is a common numerical occurrence in data fitting. Extending the dataset at both ends effectively mitigates this problem, ensuring an accurate fit within the target interval. This approach is demonstrated using data from Nancy (Figure 2). The resulting sum of sine and cosine functions is computationally straightforward to transform into the Laplace domain.

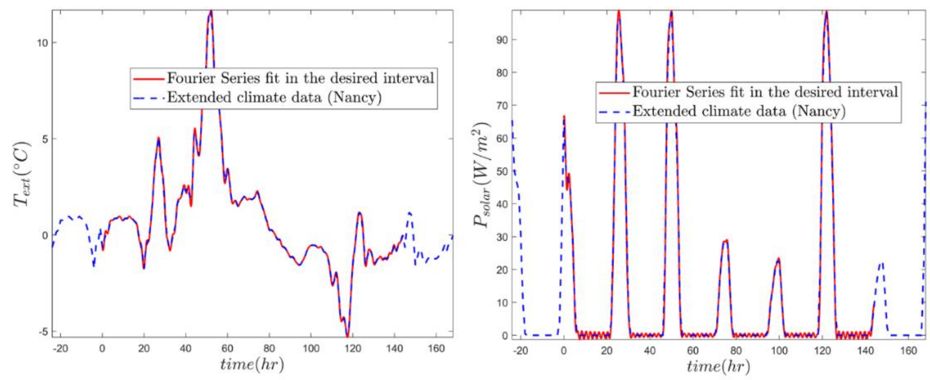



Figure 2: *Fourier series fit in the interval of interest, performed on an extended dataset to avoid the edge effects: external air temperature (left) and solar flux (right)*

### 2.2.2. Discrete Laplace transform method

This approach is not a conventional fitting technique but rather a method to represent discrete climate data in the Laplace domain. By applying this transformation, the resulting quantities become compatible with solving the quadrupole system of equations. The idea is to divide the dataset into multiple sub-series, each containing two consecutive data points. These sub-series are then summed and multiplied by a rectangular function *rect* defined over the interval $[t_i, t_{i+1}]$. This reconstruction reproduces the original dataset, as illustrated in equation (3) for temperature data:

$$T(t) = \sum_i \left( \frac{\left(T(t_{i+1}) - T(t_i)\right)(t - t_i)}{\Delta t} + T(t_i) \right) \times rect(t_i, t_{i+1}) \tag{3}$$

where $\Delta t = t_{i+1} - t_i$. The Laplace transform of the above equation is:

$$\theta(p) = \sum_i (g_i(t_i, p) - g_i(t_{i+1}, p) \tag{4}$$

with $g_i(t,p) = e^{-pt} \left( \frac{\left(T(t_{i+1}) - T(t_i)\right)t}{p\Delta t} + \frac{1}{p} \left( T(t_i) - t_i \frac{T(t_{i+1}) - T(t_i)}{\Delta t} + \frac{T(t_{i+1}) - T(t_i)}{p^2 \Delta t} \right) \right)$

To ensure continuity at the end of the series ($t = t_{end}$), an additional term is introduced in equation (4). This assumes that the temporal signal remains constant beyond $t_{end}$, maintaining the final recorded value $T(t_{end})$ till infinity. As a result, equation (4) is modified as follows:

$$\theta(p) = \frac{T(t_{end}) \times e^{-pt_{end}}}{p} + \sum_i (g_i(t_i, p) - g_i(t_{i+1}, p) \tag{5}$$

The external air temperature data for Carpentras were initially converted into the Laplace domain using this method, then numerically reconstructed in the time domain via an inverse Laplace algorithm. This process serves to validate the discrete Laplace transform approach. Figure 3 compares the recorded external air temperature over one day with the numerically retrieved data from the discrete Laplace transform.

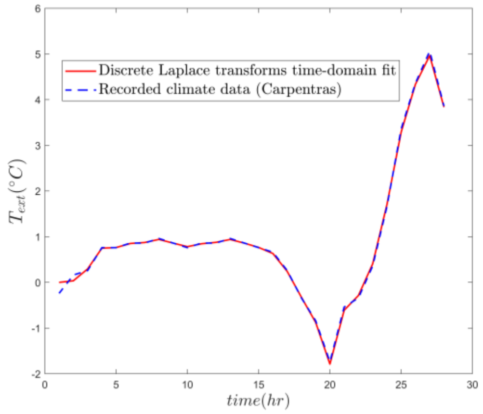



Figure 3: *Discrete Laplace transform time-domain fit versus the recorded external temperature*

### 2.3. Thermal quadrupoles applied to the 1D model

Applying the Laplace transform to the heat equation and boundary conditions of this problem, the following system of quadrupole equations is obtained:

$$\begin{pmatrix} \theta_1 \\ \phi_1 \end{pmatrix} = \left( \prod_{i=1}^{N} M_i \right) \begin{pmatrix} \theta_N \\ \phi_N \end{pmatrix} \tag{6}$$

$$\begin{bmatrix} \theta_1 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \phi_{exc} \end{bmatrix} + \begin{bmatrix} 1 & -{}^{1}/_{h_{int}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_{int} \\ \phi_{int} \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} \theta_N \\ \phi_N \end{bmatrix} = \begin{bmatrix} 0 \\ -\phi_{abs} \end{bmatrix} + \begin{bmatrix} 1 & {}^{1}/_{h_{ext}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_{ext} \\ \phi_{ext} \end{bmatrix} \tag{8}$$

where $\phi_{exc}$, $\phi_{abs}$, $\theta_{int}$, and $\theta_{ext}$ are known quantities obtained from their corresponding time-domain counterparts $\varphi_{exc}$, $\varphi_{abs}$, $T_{int}$ and $T_{ext}$, respectively. Therefore, equations (6) to (8) represent a linear system of 6 equations and 6 unknowns, namely $\phi_{ext}$, $\phi_{int}$, $\phi_1$, $\phi_2$, $\theta_1$, and $\theta_N$ which are the respective Laplace domain representations of $\varphi_{ext}$, $\varphi_{int}$, $\varphi_1$, $\varphi_2$, $T_1$ and $T_N$.

### 2.4. Laplace inversion algorithms

Retrieving the resolved quantities of the previous system of quadrupole equations in the original time domain requires applying an inverse Laplace transform, which is essentially the solution of the following integral [10]:

$$f(t) = \mathcal{L}^{-1}(F(p)) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(p)e^{pt}dp \tag{9}$$

This requires a numerical approximation, for this purpose, three algorithms are investigated to optimize the inversion process in the problem where discrete data were fitted and incorporated in the quadrupole equations: Gaver-Stehfest [11], De Hoog [12], and Den Iseger [13].

## 3. Results and discussion

### 3.1. Testing layout

Two multilayer wall configurations are investigated, the Internal Thermal Insulation wall (denoted ITI) and the Single-Wall Structure (denoted SWS). These walls are common in French buildings. Figure 4 shows an illustration of the two types; their layer composition and thermal properties can be found in [13].

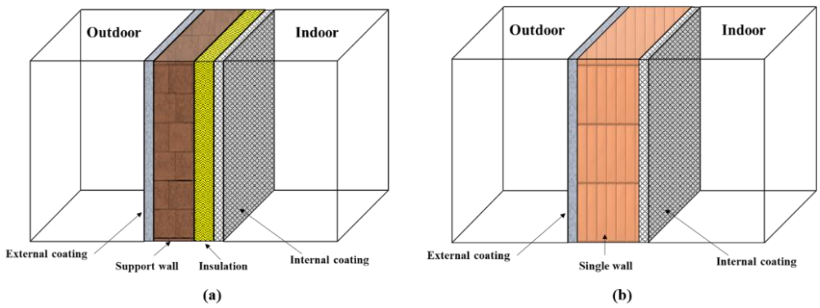



Figure 4: *(a) Internal Thermal Insulation (ITI) and (b) Single-Wall Structure (SWS) layouts*

For a total simulation duration of 6 days, the excitation $\varphi_{exc} = 400$ W.m$^{-2}$ begins at $t_{exc} =$ 73 hours. The heat equation in this problem is linear, allowing the total solution to be obtained by summing three independent solutions, each corresponding to a specific boundary condition or source term. The three solutions account for the interior air temperature, the exterior air temperature and solar flux, and the thermal excitation on the interior surface. The thermal quadrupole method represents these responses using transfer matrices, decoupling the effects of boundary conditions and enabling each contribution to be treated separately and then linearly combined. This allows also the advantage of looking at each individual solution alone depending on the desired investigation.

### 3.2. Effect of the climate data incorporation method and Laplace inversion algorithms

To study the potential of each climate data incorporation method proposed, the model results are compared against a finite element resolution of the problem with FreeFEM++. An ITI wall with the Carpentras climate is studied. Den Iseger inversion is used and the entire superposed solution is obtained. The quantities observed are the external surface temperature $T_{SE}$ (°C) and heat flux $\phi_{SE}$ (W.m$^{-2}$).

Figure 5 compares the performance of the Fourier series and discrete Laplace transform methods for incorporating climate data in the quadrupoles model, showing that both methods are in good agreement with the FreeFEM results, with minor discrepancies at the simulation edges. Although the Fourier series method exhibits more severe discrepancies, both methods effectively approximate the FreeFEM reference throughout most of the simulation, and these minor errors can be easily addressed using a numerical technique similar to the one used to avoid the edge effects.

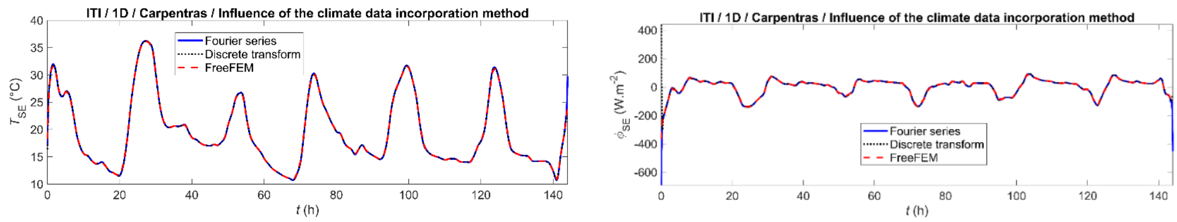



Figure 5: *Comparison of the climate data incorporation techniques*

### 3.3. Laplace inversion algorithms

Another investigation involves the three Laplace inversion algorithms presented in section 2.4. An ITI wall with the Carpentras climate is studied. Fourier series fitting is used, and the entire superposed solution is obtained.

Figure 6 presents the results of different inversion algorithms used to solve the thermal quadrupole equations and compares them with FreeFEM results for validation. The Gaver-Stehfest algorithm proves inadequate for problems with oscillations and abrupt changes, while the De Hoog algorithm offers acceptable results initially but fails over the duration of the simulation, it doesn't accommodate well the incorporation of climate data. This algorithm, however, is exceptionally fast (at least 10 times faster than other algorithms) and accurate when dealing with the subproblem where only the excitation boundary condition is considered. This provides an opportunity to couple it with the Den Iseger algorithm which delivers accurate and fast results for climate data integration, as shown in Figure 6.

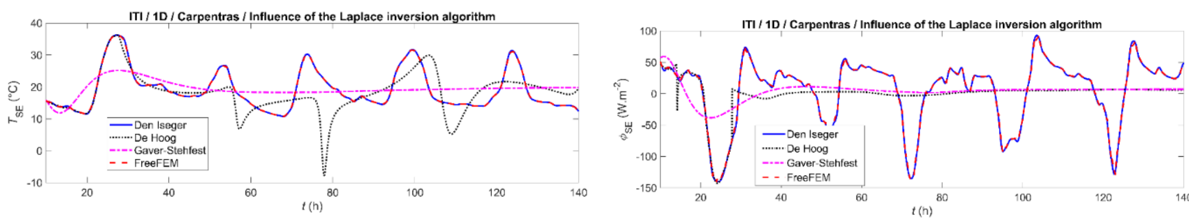
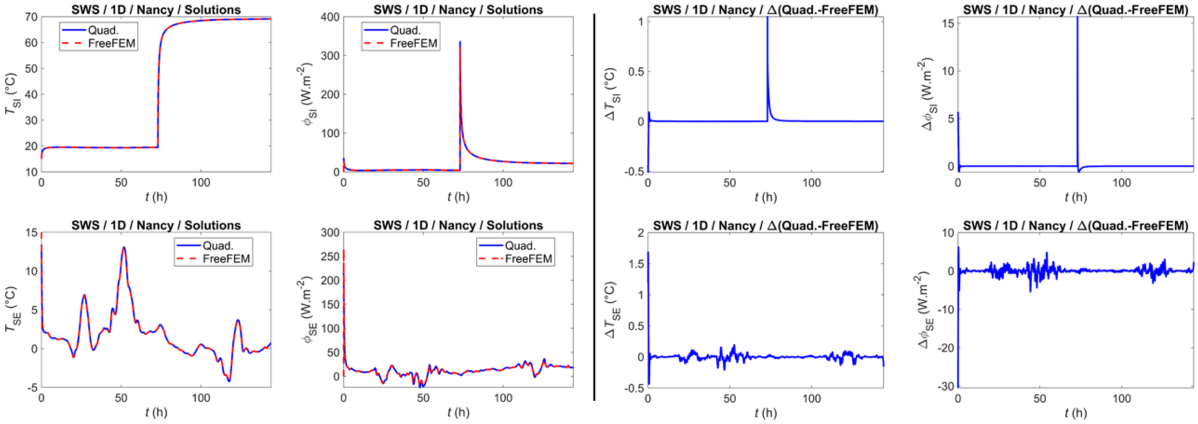



Figure 6: *Laplace inversion algorithm investigation*

### 3.4. Quadrupole model comparison with FreeFEM++

To further grasp the extent of the accuracy of the model and to evaluate the computational cost saving potential, a SWS wall is simulated with Nancy climate data, fitted with discrete Laplace transform. Den Iseger algorithm is used for the two subproblems where (1) only interior air temperature is considered and where (2) only the external air temperature and solar flux are considered. The De Hoog algorithm is used for the excitation subproblem.

Figure 7a compares the quadrupole model results with FreeFEM, the quantities of interest are the interior surface temperature $T_{SI}$ (°C) and heat flux $\phi_{SI}$ (W.m$^{-2}$), presented in the top panel. In addition to the exterior surface temperature $T_{SE}$ (°C) and heat flux $\phi_{SE}$ (W.m$^{-2}$), shown in the bottom panel. As observed in the figures, a strong agreement is achieved between the two approaches except for an obvious error at the initial edge of the graphs, likely due to the discrete Laplace transform used for climate data integration. To ensure a better comparison, Figure 7b presents the differences between the solutions, which would ideally remain at zero throughout the simulation. This reveals a mismatch in the quadrupole solution at the interior surface when the excitation begins. The discrepancies at the initial edges of the graphs are again stressed in this representation.




(a) (b)

Figure 7: *(a) Solutions of the quadrupole and FreeFEM++ models and (b) their corresponding solution differences*

Those slight discrepancies are acceptable when considering the immense trade-off in terms of computation time. The quadrupole model solves the problem in 1 second as opposed to a 25 second resolution with finite elements on FreeFEM (computation performed on a laptop with an 8-core processor and 16 GB of RAM). This is a considerable improvement in the context of a direct thermal model that will be utilized in an inverse problem in future studies.

## 4. Conclusions

This paper explores the integration of discrete climate data, such as external air temperature and solar flux, into the thermal quadrupole model of multilayered building walls. The aim is to develop a fast and reliable method for solving heat transfer problems, which could later serve as a direct model for inverse thermal property estimation. Instead of using traditional sinusoidal approximations or transfer function convolutions, the study proposes fitting climate data using either a Fourier series or a discrete Laplace transform, both of which facilitate integration into the quadrupole equations.

To solve the quadrupole model, the superposition theorem is applied, breaking the problem into three subproblems based on boundary conditions. Three inversion algorithms, namely Gaver-Stehfest, De Hoog, and Den Iseger, are tested for transforming the solution back to the time domain. Gaver-Stehfest proves unsuitable, while De Hoog performs well only for the excitation subproblem. Den Iseger demonstrates superior accuracy and is recommended for integrating real climate data. Finally, the quadrupole model significantly outperforms the finite element method in computational efficiency, achieving a 1-second resolution compared to 25 seconds in FreeFEM++. This study provides a structured approach for building an efficient thermal quadrupole model, demonstrating its potential as a fast and accurate tool for inverse heat transfer problems.

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