Identification de systèmes thermiques linéaires et non linéaires par des structures mathématiques d'intégration d'ordre non entier

Identification of linear and non linear thermal systems from non integer integral mathematical structures

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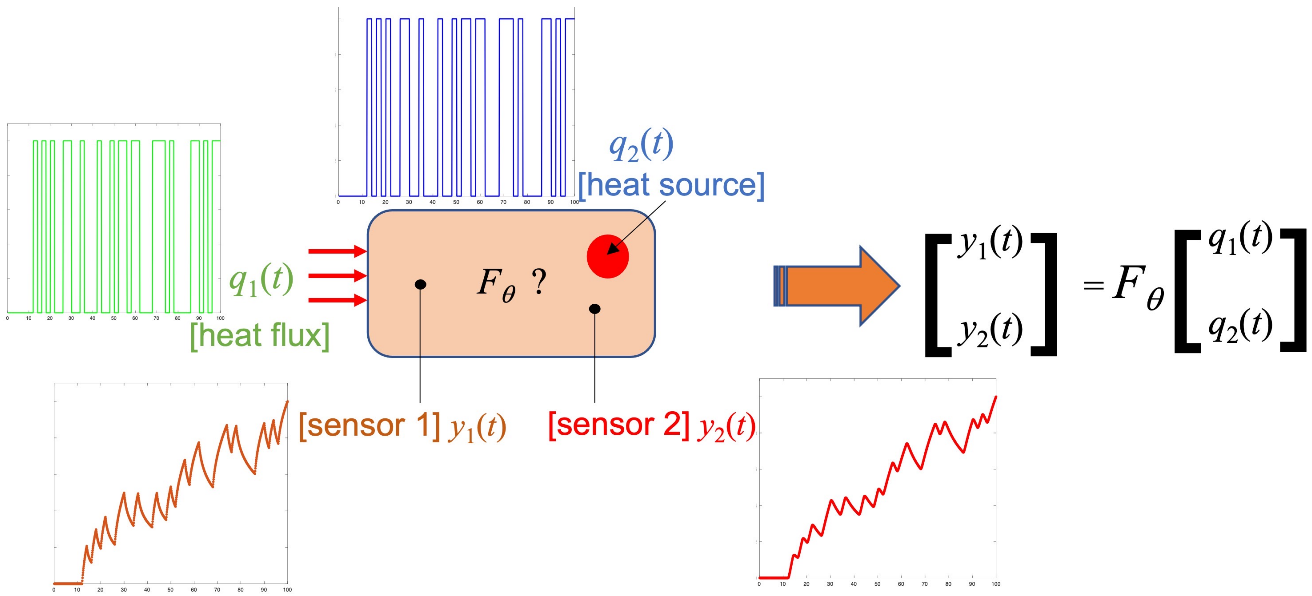
Journée SFT : Inversion de données faisant appel à un modèle en thermique, quels apports de l'intelligence artificielle ?

System identification

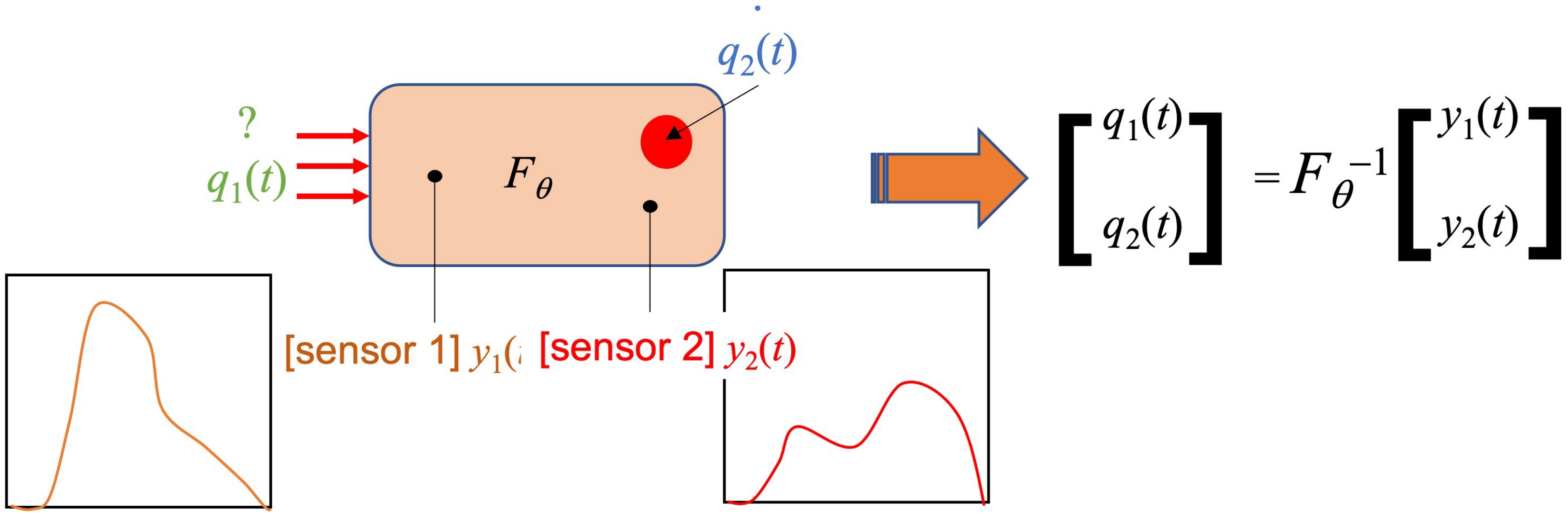


A model F_θ that relates the temperature change $y_j(t)$, $j=1,\ldots,N_c$, of sensors to the thermal BC $q_i(t)$, $i=1,\ldots,N_q$ (can be either a temperature, a heat flux or a source) is **identified** from measurements of those quantities.

The model « learns » from the data

Potential applications

- Simulation
- Control
- Estimation of $\boldsymbol{q}(t)$ (IHCP)
- …



F_θ model structure (monovariable system)

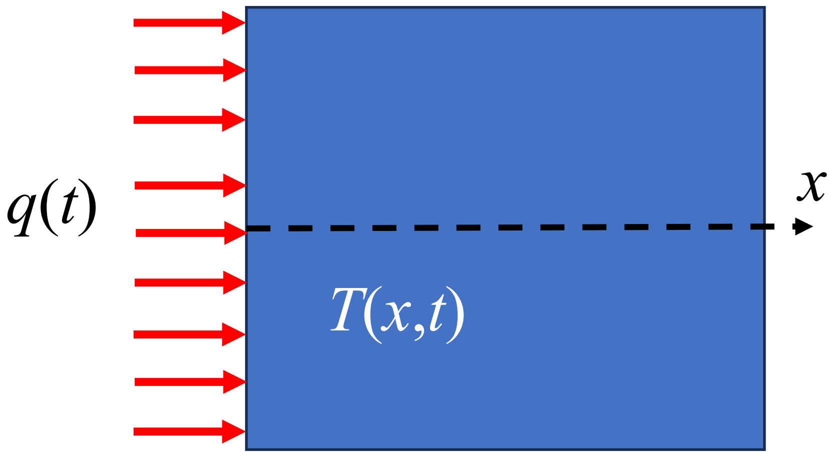
$$\sum_{k=0}^{\infty} a_k \, \mathrm{D}^k \{y(t)\} = \sum_{k=0}^{\infty} b_k \, \mathrm{D}^k \{q(t)\} + \varepsilon(t) \, , \; a_0 = 1$$

$$\mathrm{D}^k \{f(t)\} = \mathrm{d}^k f(t) / \mathrm{d}t^k$$

Using finite difference discretization for the derivatives, the relation is equivalent to the family of exogenous auto-regressive structures (AR, ARMA, ARMAX, OE, etc.)

$$\sum_{k=0}^{\infty} \alpha_k \, y(t-k) = \sum_{k=0}^{\infty} \beta_k \, q(t-k) + \varepsilon(t) \, , \; a_0 = 1$$

Heat diffusion in a semi-infinite medium



$$\theta\left(x,p\right)=H\left(x,p\right)\,Q\left(p\right)$$

$$H\left(x,p\right)=\frac{e^{-\frac{x}{\sqrt{\alpha}}\sqrt{p}}}{\lambda\sqrt{p/\alpha}}=\frac{e^{-\frac{x}{\sqrt{\alpha}}\sqrt{p}}}{E\sqrt{p}}$$

At $x=0$ $$\boxed{H\left(0,p\right)=\frac{1}{E\sqrt{p}}}$$

Let us recall that the solution is: $T\left(0,t\right)=\dfrac{1}{E\sqrt{\pi}\sqrt{t}}*q$

The non integer derivative

$$\mathscr{L}\left(\frac{\mathrm{d}^v f(t)}{\mathrm{d}t^v}\right) = s^v F(s) - \sum_{k=0}^{n-1} s^{n-k-1}\frac{\mathrm{d}^v f(0)}{\mathrm{d}t^v}$$

Liouville demonstrated that this definition remains exact when v is a real and even more generally a complex number

$$\mathrm{D}^v\{f(t)\} = \mathrm{D}^n\left\{\mathrm{I}^{n-v}\{f(t)\}\right\}, n \in \mathrm{N}, \mathrm{Re}(v) > 0, n-1 \leq \mathrm{Re}(v) < n$$

Non-integer derivative

$$\mathrm{I}^v\{f(t)\} = \frac{1}{\Gamma(v)}\int_0^t (t-u)^{v-1} f(u)\mathrm{d}u, \mathrm{Re}\,(v) > 0$$

Non-integer integral

Convolution product of $f(t)$ with t^{v-1} (infinite memory)

$$\Gamma(v) = \int_0^\infty u^{v-1}\exp(-u)\,du$$

New expression of the solution using the non-integer model structure

$$\mathscr{L}^{-1}\left(\frac{1}{\sqrt{p}}\,F\left(p\right)\right)=\mathrm{I}^{1/2}\{f(t)\}$$

If one wants to use the integer structure, an infinite number of terms must be considered

$$H\left(0,p\right)=\frac{1}{E\,\sqrt{p}}\quad\Rightarrow\quad T\left(0,t\right)=\frac{1}{E}\,\mathrm{I}^{1/2}q$$

$$T\left(x,t\right)=\sum_{n=0}^{\infty}b_n\mathrm{I}^{n/2}\left\{q\left(t\right)\right\}$$

Optimal model structure for this configuration:

$$y=\beta_1\,\mathrm{I}^{1/2}q+\varepsilon$$

$$\beta_{1,\mathrm{th}}=1/E$$

How to calculate the non integer integral ?

Several discretization schemes (Grünwald for instance)

$$D^{\nu} f(t_0) \sim \frac{1}{h^{\nu}} \sum_{k=0}^{n-1} (-1)^k \binom{\nu}{k} f(t_0 - kh), \qquad \nu > 0 \qquad\qquad \binom{\nu}{k} = \frac{\nu\,(\nu-1)\cdots(\nu-k+1)}{k!}$$

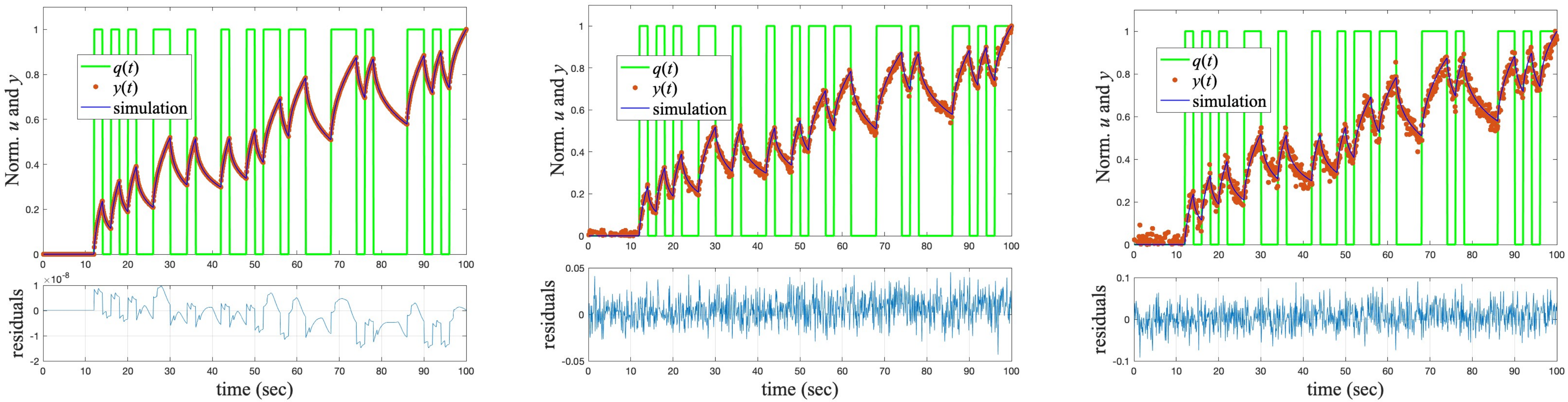
Another way:

$$\mathrm{I}^{v}\{f(t)\} = \frac{1}{\Gamma(v)} \int_0^t (t-u)^{v-1} f(u)\mathrm{d}u, \mathrm{Re}\,(v) > 0 \;\Rightarrow\; \mathrm{I}^{\zeta}\{f(t)\} = \mathscr{L}^{-1}\left(\frac{1}{p^{\zeta}}\right) * f(t) = \frac{t^{\zeta-1}}{\Gamma(\zeta)} * f(t),\ \zeta \geq 0$$

FFT is fast

$$\boxed{\frac{t^{\zeta-1}}{\Gamma(\zeta)} * f(t) = \mathrm{FFT}^{-1}\left[\mathrm{FFT}\left[\frac{t^{\zeta-1}}{\Gamma(\zeta)}\right] \times \mathrm{FFT}\left[f(t)\right]\right]}$$

Illustration



$$y\,(t) = T\,(0,t) + \varepsilon\,N\,(t)$$ (simulated from the exact solution)

Noise amp.	$\beta_{1,\text{id}}$ (LS, RLS)	$\beta_{1,\text{th}} = 1/E$
$\varepsilon = 0$	$1.26 \times 10^{-4} \pm 1.82 \times 10^{-14}$	1.26×10^{-4}
$\varepsilon = 5\%$	$1.26 \times 10^{-4} \pm 1.0 \times 10^{-7}$	1.26×10^{-4}
$\varepsilon = 10\%$	$1.26 \times 10^{-4} \pm 2 \times 10^{-7}$	1.26×10^{-4}

The parameter identification is based on the linear least squares (recursive RLS or not LS)

What happens if the sensor is located at $x > 0$

$\exp(z) = \sum_{n=0}^{\infty} z^n / n!$ ⇒ $$H(x,p) = \sum_{n=0}^{\infty} \frac{\left(-\frac{x}{\sqrt{\alpha}}\sqrt{p}\right)^n}{n!\,E\,\sqrt{p}} = \frac{1}{E\,\sqrt{p}} \left(1 + \sum_{n=1}^{M\to\infty} \frac{-x^n}{\alpha^{n/2}\,n!}\,p^{n/2}\right)$$

$\boxed{x = 1\,\mathrm{mm}}$ $\boxed{M = 2}$

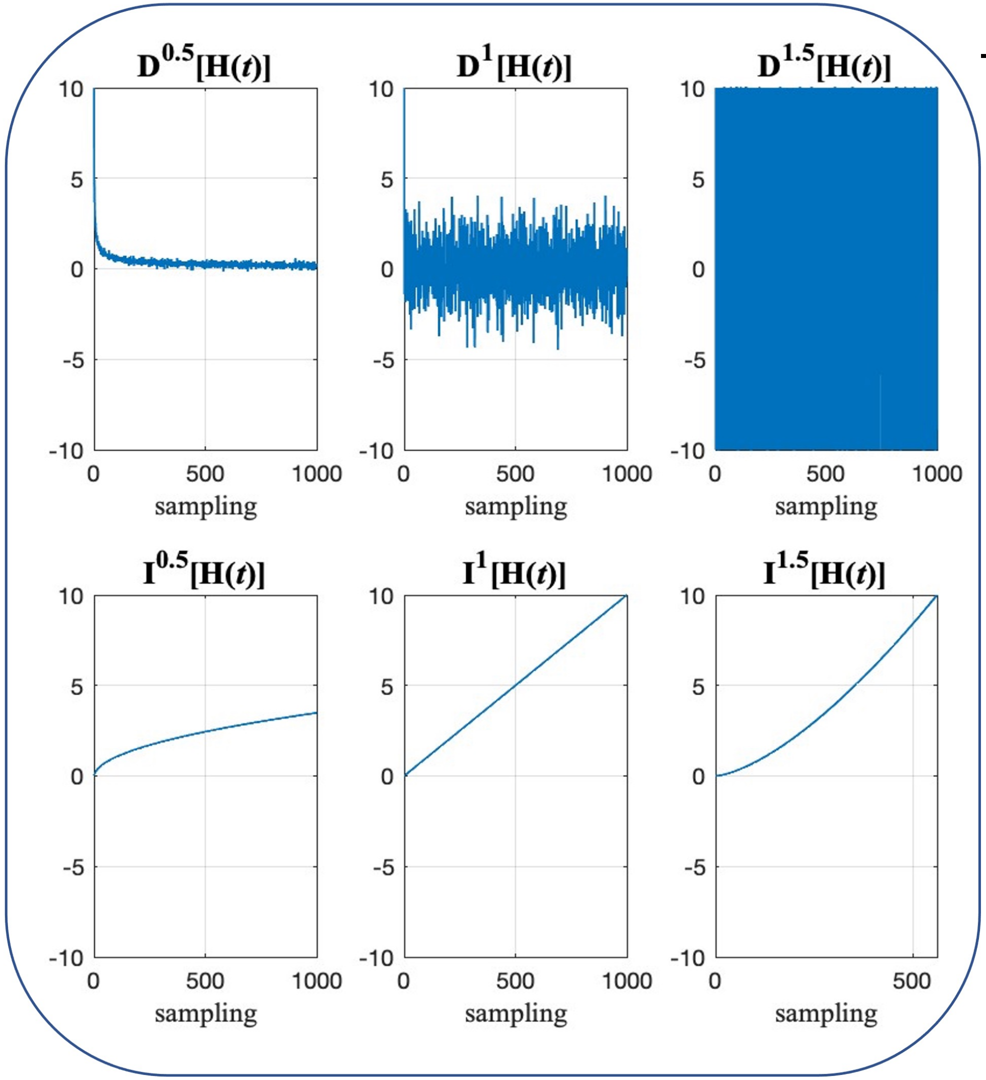
$$H(x,p) = \frac{1}{\sqrt{p}}\left(b_0 + b_1\,p^{1/2} + b_2\,p\right), \qquad b_0 = \frac{1}{E},\ b_1 = -\frac{x}{\sqrt{\alpha}\,E},\ b_2 = \frac{x^2}{2\,\alpha\,E}$$

$$H(x,p) = \frac{b_0\,p^{-1} + b_1\,p^{-1/2} + b_2}{p^{-1/2}}$$ ⇒ $$\boxed{\mathrm{I}^{1/2} y = \beta_0\,\mathrm{I}q + \beta_1\,\mathrm{I}^{1/2} q + \beta_2\,q + \mathrm{I}^{1/2}\varepsilon}$$

Why using the NI integral instead of the NI derivative ?

$U(t) = H(t)$ is the Heaviside function

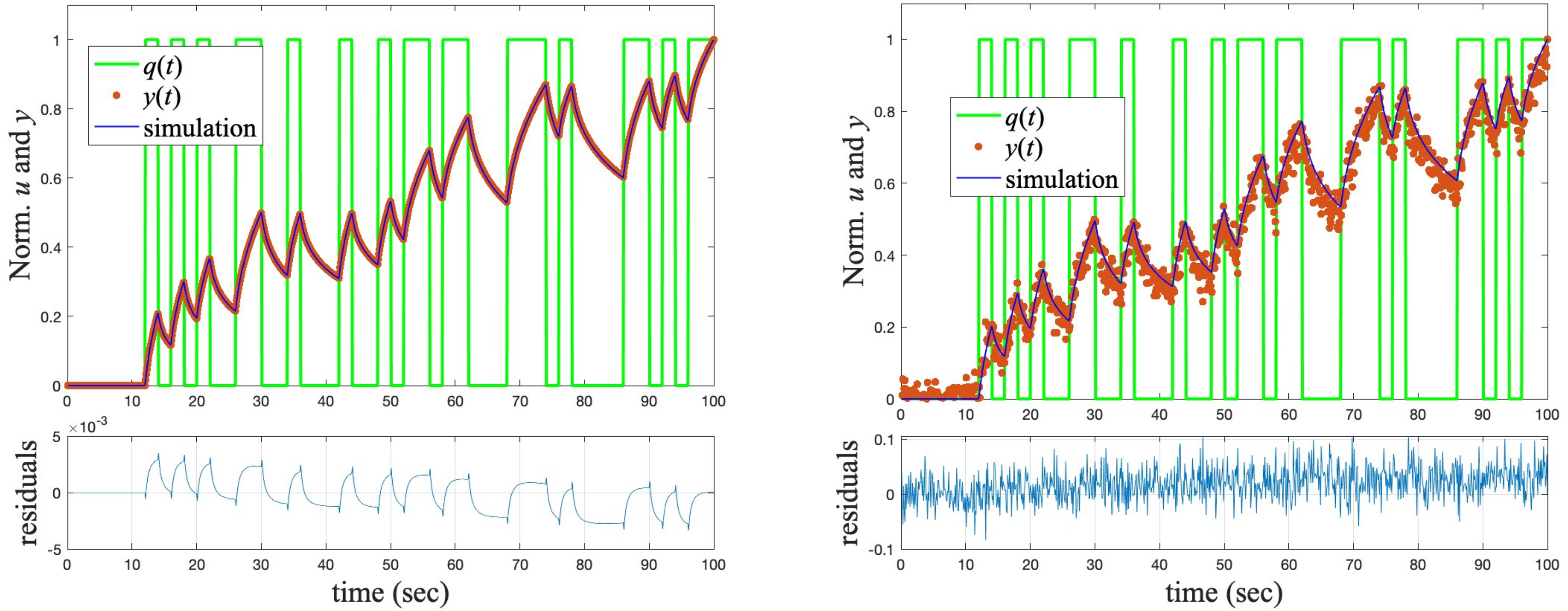
$$U_\varepsilon(t) = U(t) + \varepsilon N(t)$$



$\varepsilon = 0.01$

$\varepsilon = 0$

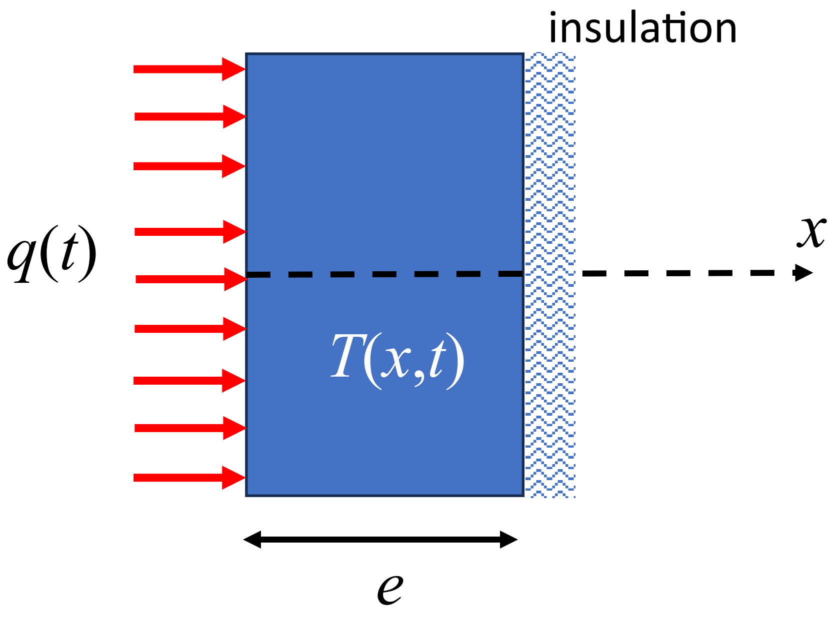
Illustration



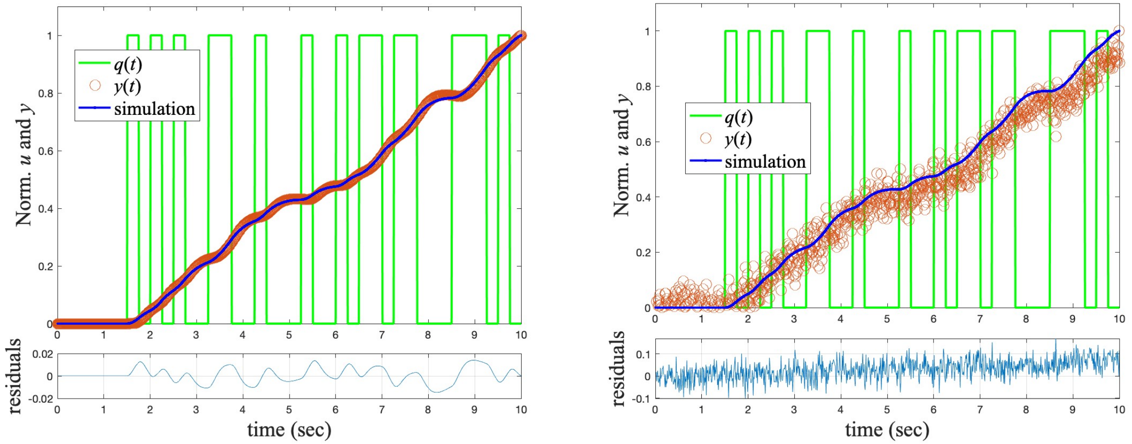
$$y\,(t) = T\,(0,t) + \varepsilon\,N\,(t)$$ (simulated from the exact solution)

Noise amp.	β_0	β_1	β_2
theoretical values	1.261×10^{-4}	-6.67×10^{-5}	1.76×10^{-5}
id. values, $\varepsilon = 0$	$1.255 \times 10^{-4} \pm 1.67 \times 10^{-5}$	$-9.523 \times 10^{-5} \pm 1.27 \times 10^{-4}$	$1.3 \times 10^{-5} \pm 1.67 \times 10^{-4}$
id. values, $\varepsilon = 10\%$	$1.265 \times 10^{-4} \pm 1.65 \times 10^{-5}$	$-10.37 \times 10^{-5} \pm 1.27 \times 10^{-4}$	$1.7 \times 10^{-5} \pm 1.66 \times 10^{-4}$

Loss of the semi-infinite behaviour



$x = 3\text{mm}$, $e = 1$ cm

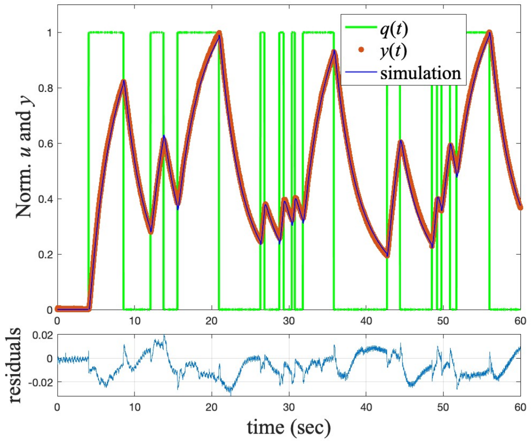


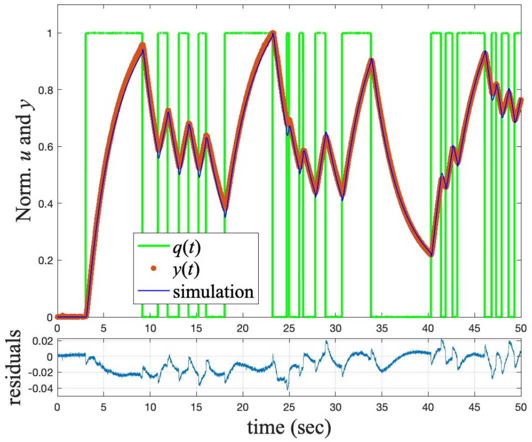
The best optimal structure is: $$y + \alpha_1 \, \mathrm{I}y = \beta_0 \, \mathrm{I}^2 q + \beta_1 \, \mathrm{I}^3 q + \beta_2 \, \mathrm{I}^4 q$$

Can be derived from the exact solution of the tranfer function: $$H(x,p) = \frac{\cosh\left(\sqrt{p/\alpha}\,(e-x)\right)}{\lambda\sqrt{p/\alpha}\,\sinh\left(\sqrt{p/\alpha}\,e\right)}$$

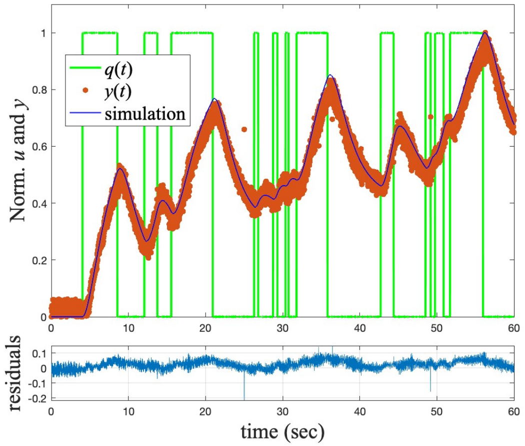
Real application

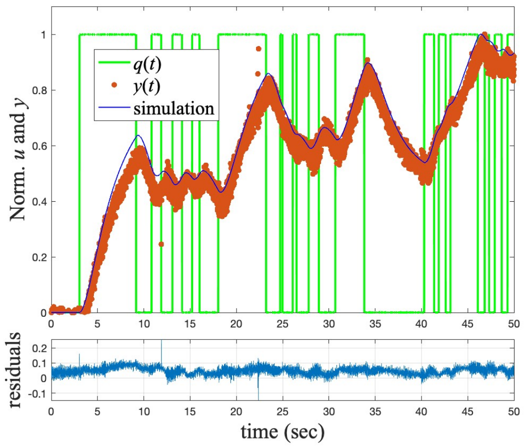
System identification for Sensor 1





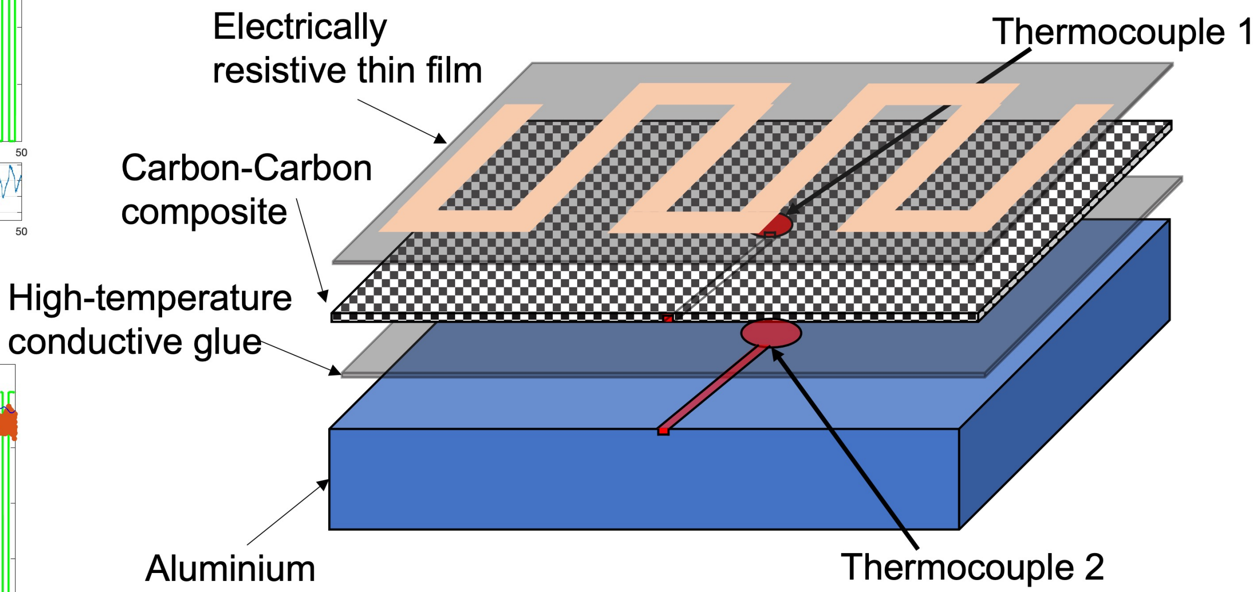
System identification for Sensor 2





Learning

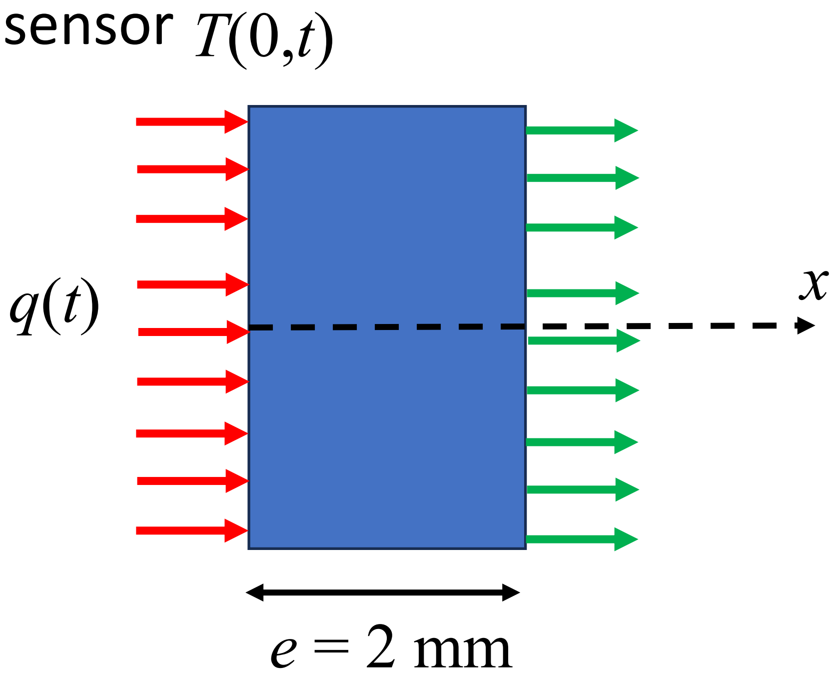
validating

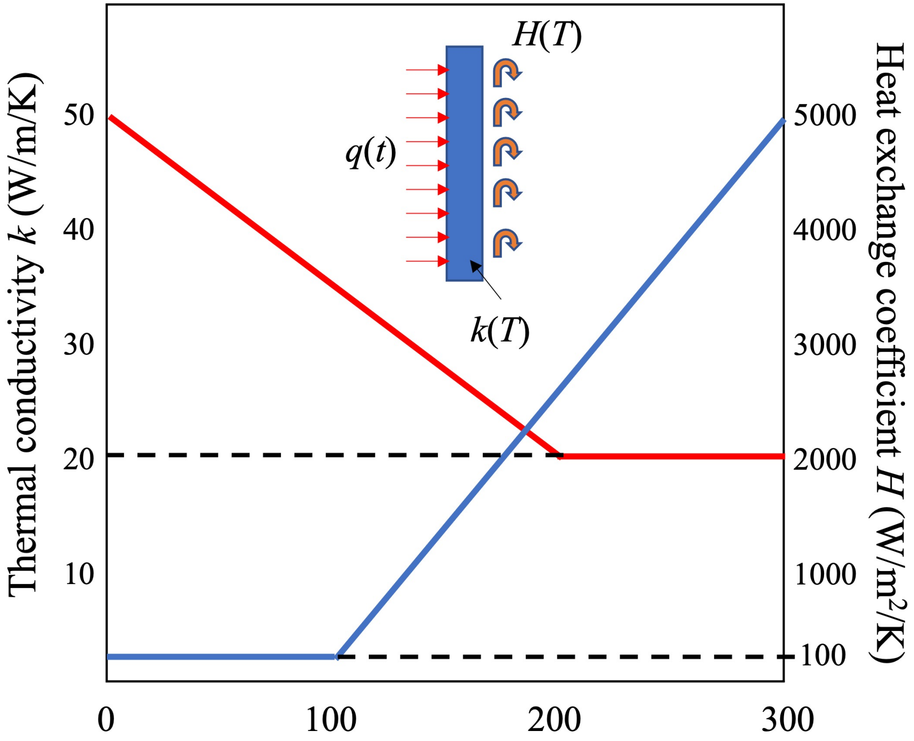


Identified models

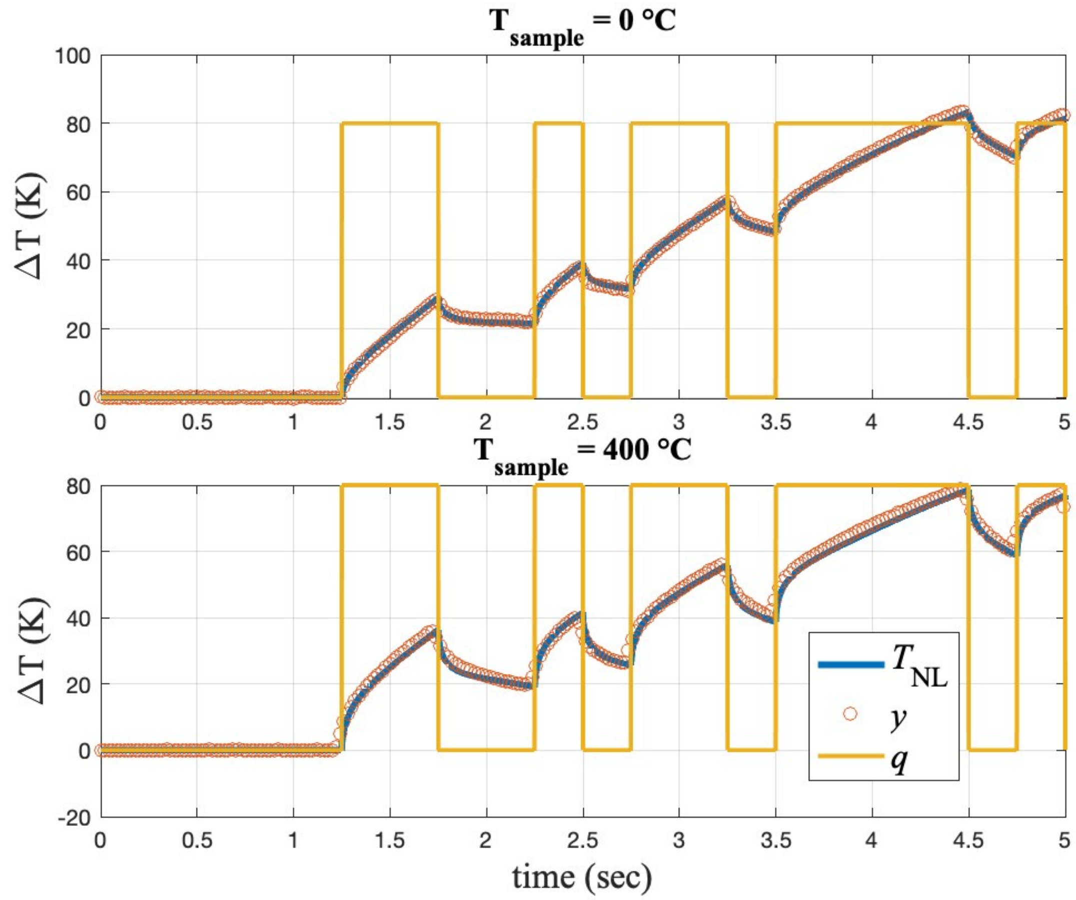
<table>
<tr><td colspan="8">$y_1 = \beta_1\,\mathrm{I}^{0.5}q + \beta_2\,\mathrm{I}^{1}q + \beta_3\,\mathrm{I}^{1.5}q + \beta_4\,\mathrm{I}^{2}q + \beta_5\,\mathrm{I}^{2.5}q + \beta_6\,\mathrm{I}^{3}q + \beta_7\,\mathrm{I}^{3.5}q + \varepsilon\,(t)$</td></tr>
<tr><td>parameter</td><td>β_0</td><td>β_1</td><td>β_2</td><td>β_3</td><td>β_4</td><td colspan="2">β_5</td></tr>
<tr><td>value</td><td>-0.9</td><td>3.04</td><td>-1.92</td><td>0.61</td><td>-0.1</td><td colspan="2">0.007</td></tr>
<tr><td>std</td><td>0.004</td><td>0.006</td><td>0.005</td><td>0.002</td><td>0.0004</td><td colspan="2">3.5×10^{-5}</td></tr>
<tr><td colspan="8">$y_1 + \alpha_1\,\mathrm{I}^{1}y + \alpha_1\,\mathrm{I}^{2}y = \beta_1\,\mathrm{I}^{2}q + \beta_2\,\mathrm{I}^{3}q + \beta_3\,\mathrm{I}^{4}q + \beta_4\,\mathrm{I}^{5}q + \beta_5\,\mathrm{I}^{6}q + \varepsilon\,(t)$</td></tr>
<tr><td>parameter</td><td>α_1</td><td>α_2</td><td>β_1</td><td>β_2</td><td>β_3</td><td>β_4</td><td>β_5</td></tr>
<tr><td>value</td><td>1.31</td><td>0.45</td><td>0.13</td><td>0.007</td><td>3.2×10^{-5}</td><td>-8.5×10^{-6}</td><td>2.67×10^{-7}</td></tr>
<tr><td>std</td><td>0.021</td><td>0.004</td><td>0.0017</td><td>0.0003</td><td>2.55×10^{-5}</td><td>1.84×10^{-6}</td><td>6.27×10^{-8}</td></tr>
</table>

The case of non-linear heat diffusion

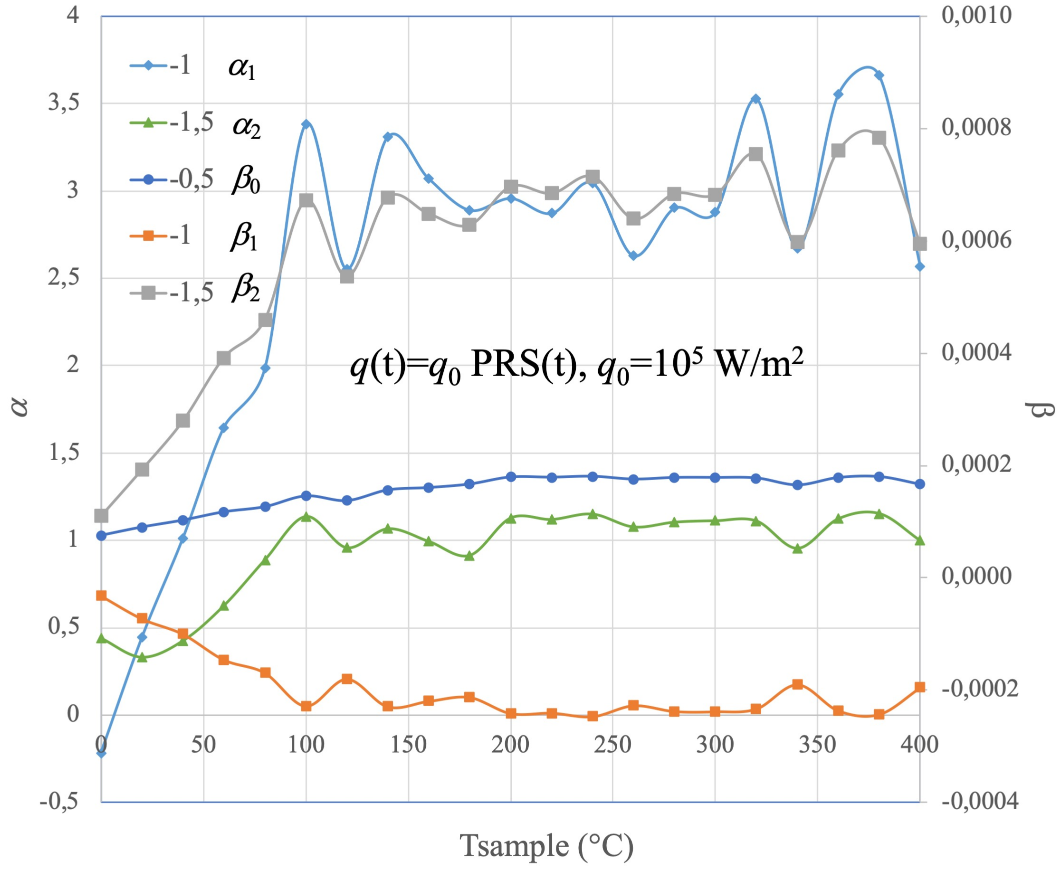




System identification



$$T + \alpha_1 \, \mathrm{I}^1 q + \alpha_2 \, \mathrm{I}^{1.5} q = \beta_0 \, \mathrm{I}^{0.5} q + \beta_1 \, \mathrm{I}^1 q + \beta_2 \, \mathrm{I}^{1.5} q$$



Justification

Volterra series decomposition

$$T(t) = \sum_{j=1}^{\infty} T_j(t) = T_1(t) + T_2(t) + \cdots + T_j(t) + \cdots$$

$$T_j(t) = \int_0^{\infty} \cdots \int_0^{\infty} h_j(\tau_1, \cdots, \tau_j) \prod_{i=1}^{j} q(t-\tau_i)\, \mathrm{d}\tau_i$$

j^{th} order Volterra kernel

In practice the number of kernels can be limited to 2

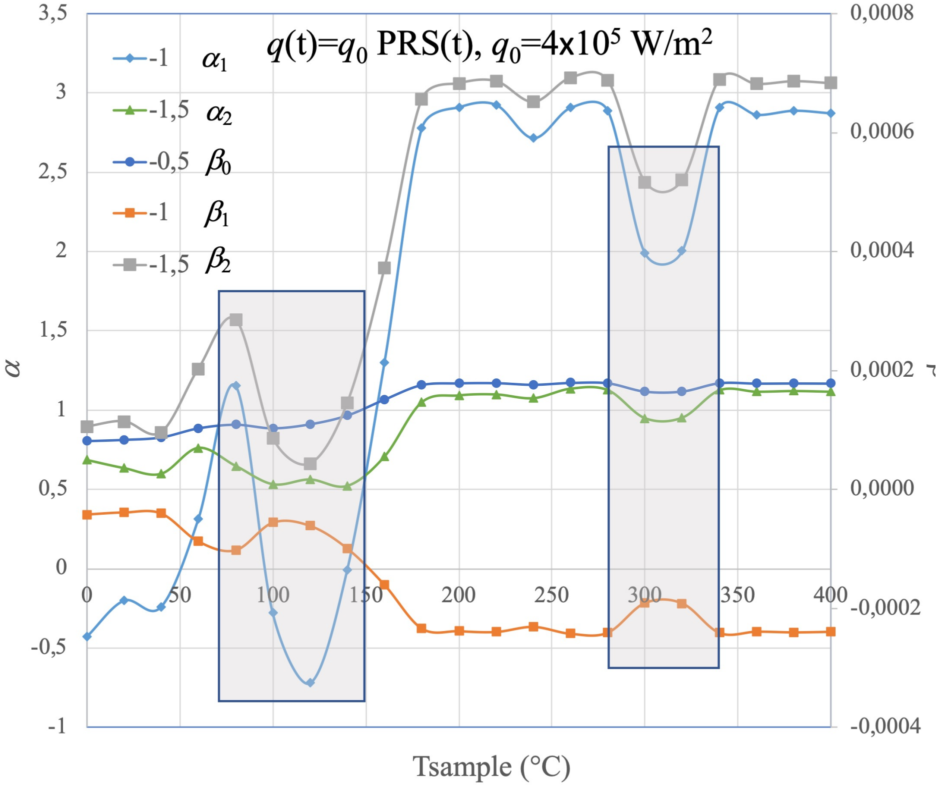
$$T(t) = T_1(t) + T_2(t) = T_L(t) + T_{NL}(t)$$
$$= \int_0^{\infty} h_1(\tau)\, q(t-\tau)\, \mathrm{d}\tau + \int_0^{\infty}\int_0^{\infty} h_2(\tau_1,\tau_2)\, q(t-\tau_1)\, q(t-\tau_2)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2$$

We propose to introduce a **generalized transfer function**:

$$\boxed{T(t) = \int_0^{\infty} h_{NL}(\tau)\, q(t-\tau)\, \mathrm{d}\tau}$$

Justification

Since the non integer integral involves all the time history of the integrand function, it is well suited with the introduction of a generalized transfer function that accounts with non-linearities.



Increasing q_0 leads to involve larger non linearities and thus to limit the range of temperature investigated by the model

$$T + \alpha_1 I^1 q + \alpha_2 I^{1.5} q = \beta_0 I^{0.5} q + \beta_1 I^1 q + \beta_2 I^{1.5} q$$