L8: Experimental modelling through identification of low order models

Jean-Luc BATTAGLIA

TREFLE-I2M, Esplanade des Arts et Métiers, 33405 Talence, France

E-mail: jean-luc.battaglia@bordeaux.ensam.fr

Abstract

The system identification technique is used in order to formulate a reliable direct model to be used in an inverse heat transfer problem. This approach found several practical applications in thermal sciences for raisons that will be developed in the text. For clarity, we will restrict our presentation to monovariable linear systems relating the temperature at one point in the system to one heat flux acting on the system. Two approaches are presented in this course. In the first one, the non parametric method only used the temperature and heat flux measurement by calculating the cross correlation or power spectral density. The second set of methods relates to the parametric methods that consist in identifying the parameters of a model that expressed the successive time derivatives of the temperature to the heat flux.

Nomenclature

a	Thermal diffusivity m^2.s^{-1}	S_{xy}	power spectral density between x and y
C_{xy}	correlation function between x and y	T	temperature, K
C_p	specific heat, J kg^{-1} K^{-1}	T	time, s
D^ν	derivative of real order ν	$X_s = \begin{bmatrix} x_s & y_s \end{bmatrix}$	sensor coordinates
e	measurement error	y	temperature measurement, K
h_m	impulse response	V	Loss function
h	exchange coefficient, W m^{-2} K^{-1}	Δt	Sampling time
H	transfer function	φ	heat flux density W m^{-2}
I^ν	integral of real order ν	ρ	density, kg m^{-3}
k	thermal conductivity, W m^{-1} K^{-1}		

1 Introduction

The system identification framework is a well known domain that has applications in automatic (for control purpose mainly) and in signal processing [1][2]. For several years the heat transfer scientific community found very interesting applications of those methods for the modelling of heat and mass processes that occur in thermal systems [6][7][8]. In this course we present the system identification technique as an efficient tool in order to formulate a reliable direct model that can be used to solve the corresponding inverse heat transfer problem. In case of a *monovariable system*, as that represented in Figure 1, the inverse procedure will consist in estimating the heat flux acting on the studied system from temperature measurement at one point in the system. Let us highlight now that the methods that will be present below can be obviously generalized to multivariable systems (several heat flux or heat sources acting on a system equipped with several sensors). As an additional constraint, we will also restrict the presentation of the methods to *linear systems*. It means that the thermal properties of the system will not depend on temperature. However, system identification has been developed for non linear systems but mathematical derivations of such techniques are largely beyond the scope of this course.
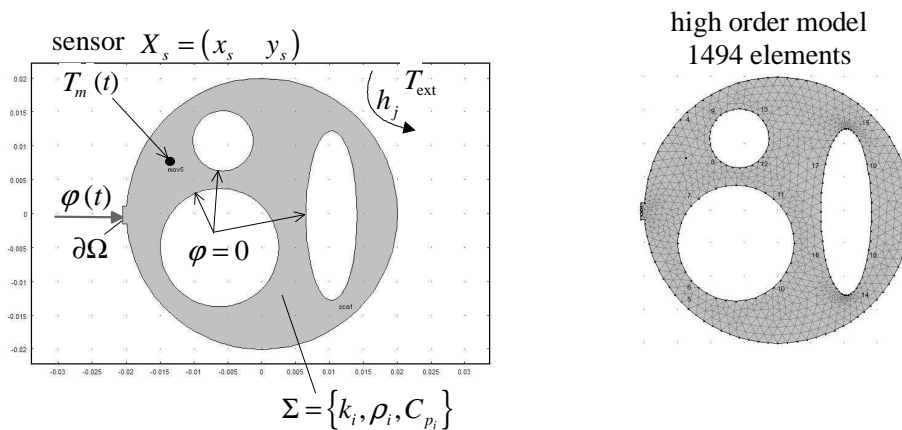

Figure 1: example of a 2D monovariable linear system.

Why scientists working in the field of heat transfer and more particularly in measurements inversion are interested with system identification? The first answer relates to model reduction. Indeed, whatever the implemented inverse technique, inversion requires simulating a direct model in an iterative manner to approach the solution. Statistical methods as the Bayesian technique one calls upon the direct model a huge number of times and computational times could become dramatically long. As an example, let us consider the 2D system represented in Figure 1. The domain Σ is characterized from its thermal properties (thermal conductivity k_i, specific heat per unit volume C_{p_i} an density ρ_i). A heat flux density φ is imposed on the boundary $\partial\Omega$ whereas the remain part of the outdoor boundary is subjected to convection with the coefficient h_j and the temperature of the surrounding fluid is

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denoted T_{ext}. Finally, the inner boundaries are insulated. The objective here is to estimate the heat flux density from temperature measurements in the plate. It is thus assumed that a sensor has been embedded in the plate and the temperature of the sensor is denoted $T_m(t)$. Although this problem is quite simple, only a discrete method (finite elements for example) can be used to solve the heat diffusion equation and associates boundary and initial conditions in order to simulate the temperature of the sensor. A mesh is thus built (see Figure 1) that leads to calculate the temperature at each node. This discrete model is so-called a *high-order model*, the order referring to the mesh degrees of freedom. Simulating this model leads to results as those presented in Figure 2.
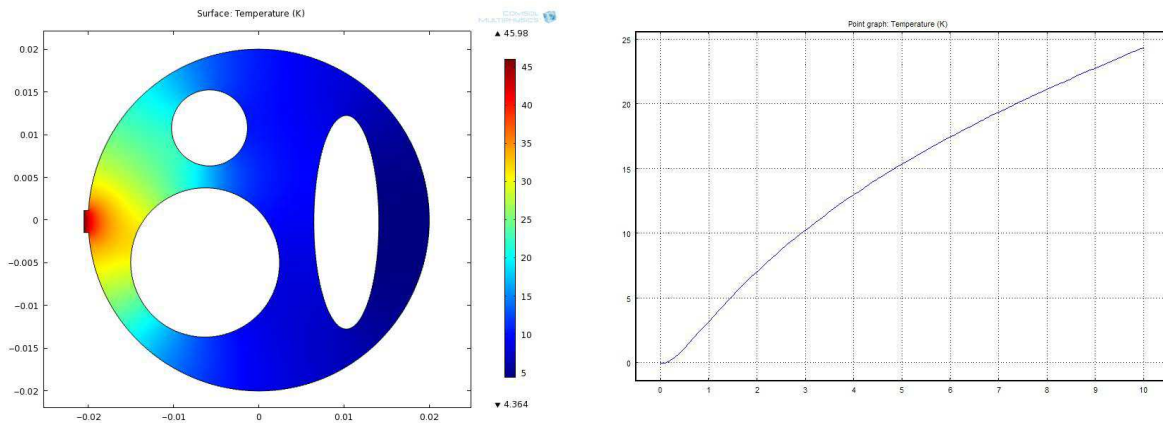

Figure 2: simulation of the temperature field at *t*=10 sec and of the time dependent temperature of the sensor for a step heat flux density.

The reliability of the direct model rests on the accuracy on two sets of data: the thermal properties $\{k_i, C_{p_i}, \rho_i, h_j\}$ and the location $X_s = [x_s \quad y_s]$ of the sensor. Uncertainties on those data will lead to a very low confidence domain for the estimated heat flux [9].

This system identification approach is described in a schematic way in Figure 3. The goal is to apply a known heat flux $\varphi(t)$ on the system and to measure the signal at the thermal sensor. We must note as a first point that it is not require *calibrating the sensor* (the link between the measured signal and the absolute temperature) since the same sensor is used both for the identification system and the inversion. Given to those data it is then possible to estimate "a" model **M** that relates them. However, it must be emphasized that this estimated model has only significance on the measurement time-domain. *Prediction* is therefore a main issue of system identification. Secondly, the measurements are affected by an error (noise) that will have an influence on the identified model. It is generally admitted that the imposed heat flux is generally fully known and that it is errorless. Thus, all the error is reported on the sensor signal.

Obviously the objective is to have the model M that is more accurate than that obtained from the FEM with uncertainties on $\left\{ k_i, C_{p_i}, \rho_i, h_j \right\}$ and $X_s = \begin{bmatrix} x_s & y_s \end{bmatrix}$.
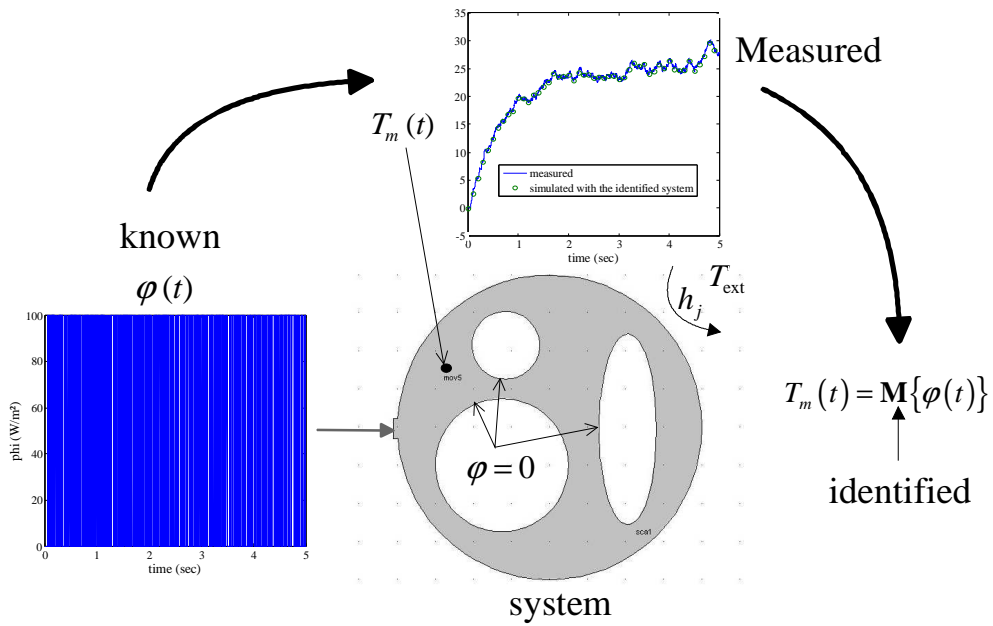

Figure 3: thermal system identification procedure.

Once the thermal system has been identified, it can be used in order to solve the inverse problem, which is to estimate the heat flux from model **M** and temperature measurement at the sensors. The classical procedure is described in Figure 4.
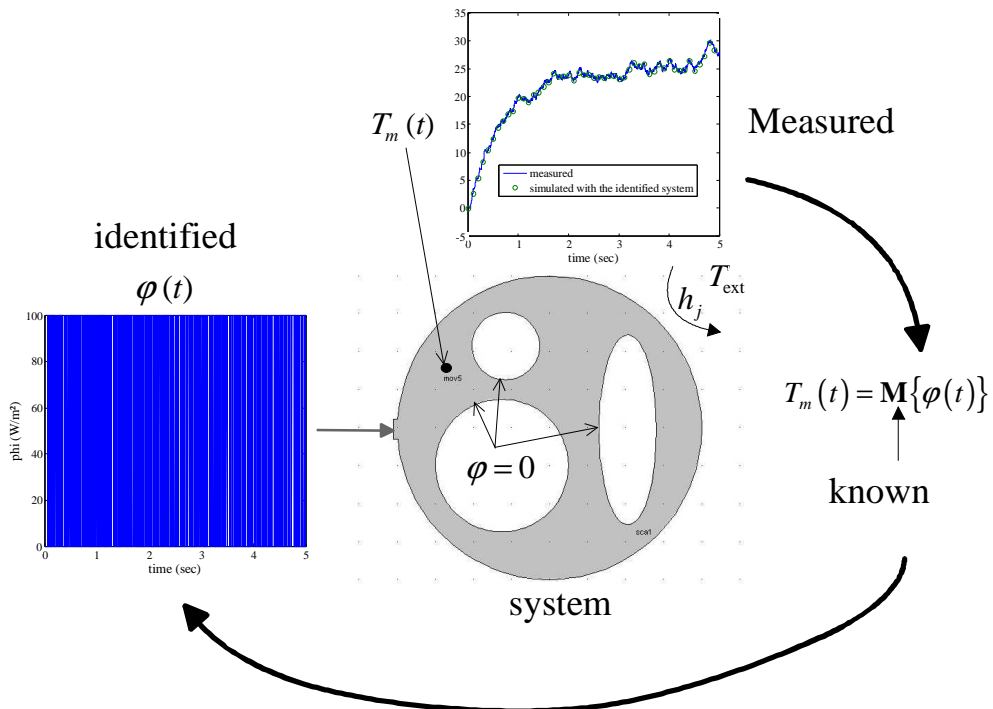

Figure 4: use of the identified system to solve the inverse procedure (estimating the heat flux).

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It means that if the identified system described well the thermal behaviour for the heat flux sequence represented in Figure 3, it is then expected to retrieve this sequence applying an inverse technique from the identified model **M** and temperature measurement represented in Figure 3. This is what suggests Figure 4.

According to our previous description, it can be thus possible now to drawn the main advantages and drawbacks of this approach.

Advantages

- The system identification approach will be first interesting to obtain a reliable and accurate *low order model* that will require less computational time for simulation.
- There is no need to know the thermal properties of the system (thermal conductivity, density, specific heat, heat exchange coefficients, thermal resistances at the interfaces, parameters related to thermal radiation…).
- It is not required to know the sensor location inside the system.
- It is not required calibrating the sensor.
- The identification procedure is fast (this will be viewed later with the description of the different techniques).

Drawbacks

- The model identification must be achieved in the exactly same conditions as those encountered during the inversion (heat exchanges between the surrounding and the system must remain the same for the two configurations).
- The prediction of the identified model rests on strong assumptions (in particular, it is better reaching the stationary behaviour during the system identification process). In general, the identified system is only valid for the time duration of the system identification process.

2 The system identification approach

2.1 The impulse response

The temperature $T_m(t)$ of the sensor is related to the heat flux density $\varphi(t)$ thanks to the impulse response $h_m(t)$ on the form of the following convolution product that is a direct mathematical formulation of the Duhamel's theorem:

$$T_m(t) = h_m(t) * \varphi(t) = \int_0^\infty h_m(t-\tau)\varphi(\tau)\,\mathrm{d}\tau \tag{1}$$

For monovariable linear systems, the impulse response fully characterizes the thermal behaviour. Therefore, any kind of inverse strategy can be based on the direct model expressed as the impulse response of the system. However, as we said in the first section, this response will depend on the

following quantities: $\{k_i, C_{p_i}, \rho_i, h_j\}$ and $X_s = \begin{bmatrix} x_s & y_s \end{bmatrix}$. According to the uncertainty that affects those quantities, the user could imagine measuring directly the impulse response from an experiment. It will consist in replacing the heat flux on the real problem by a *known* photothermal excitation, as a laser for example, and to measure the temperature of the sensor when the heat flux is delivered as a pulse. However, this approach is not reliable since the impulse response magnitude is very low, especially when one wants to preserve the linear behaviour of the system. As an illustration it is calculated the temperature of the sensor for the previous studied configuration with $\varphi = 10^6 \times \exp\left(-t^2/\tau^2\right)$ where $\tau = 1\mu\sec$ is small enough to consider the excitation as a Dirac function. The simulation is presented in Figure 5. The maximum amplitude of the response is very low and it must considered additional further impact of the measurement error.
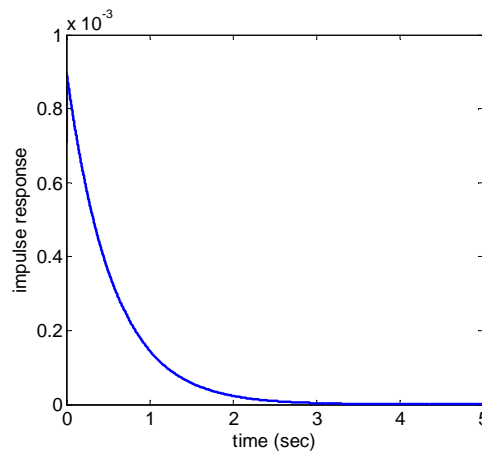

Figure 5: simulation of the impulse response using the FEM.

Another solution could consist in derivating the step response represented in Figure 2 (at the right) to retrieve the impulse response. Again, it is not a reliable technique since the derivation will amplify the measurement error and will lead to a very inaccurate impulse response, especially at the short times. Several powerful techniques have been developed in the system identification and signal processing domains that lead to more accurate impulse response of the system. These techniques are classified in two sets of methods: the non parametric methods and the parametric ones.

2.1 The non parametric approach

2.1.1 The deconvolution technique

A very easy technique for the deconvolution of (1) is to consider the discrete form of this relation [2]:

$$T_m(k\,\Delta t) = \sum_{i=0}^{k} h_m\left((k-i)\Delta t\right)\varphi(k\,\Delta t) = \sum_{i=0}^{k} h_m(k\,\Delta t)\varphi\left((k-i)\Delta t\right) \tag{2}$$

Assuming the duration of the experiment is $t_f = N \Delta t$, where Δt is the sampling time interval, relation (2) can be expressed on the form:

$$
\begin{bmatrix} T_0 \\ T_1 \\ \vdots \\ T_N \end{bmatrix} = \begin{bmatrix} \varphi_0 & & & \\ \varphi_1 & \varphi_0 & & \\ \vdots & \ddots & \ddots & \\ \varphi_N & \cdots & \varphi_1 & \varphi_0 \end{bmatrix} \begin{bmatrix} h_{m0} \\ h_{m1} \\ \vdots \\ h_{mN} \end{bmatrix}
\tag{3}
$$

With $T_k = T(k \Delta t)$ and $\varphi_k = \varphi(k \Delta t)$. Assuming an additive measurement error of normal distribution (zero mean and constant standard deviation), the measurement temperature is expressed from the real one as:

$$
y_m(k \Delta t) = T_m(k \Delta t) + e(k \Delta t) = \sum_{i=0}^{k} h_m(k \Delta t) \varphi((k-i)\Delta t) + e(k \Delta t)
\tag{4}
$$

Given that $\lim_{k \to \infty} h_k = 0$, it is reasonable to truncate the series from $k = Q$ and thus relation (3) becomes:

$$
\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_Q \\ \vdots \\ y_N \end{bmatrix}}_{Y_N} = \underbrace{\begin{bmatrix} \varphi_0 & & & \\ \varphi_1 & \varphi_0 & & \\ \vdots & \ddots & \ddots & \\ \varphi_Q & \cdots & \varphi_1 & \varphi_0 \\ \vdots & \cdots & \vdots & \vdots \\ \varphi_N & \cdots & \varphi_{N-Q+1} & \varphi_{N-Q} \end{bmatrix}}_{\Phi_N} \underbrace{\begin{bmatrix} h_{m0} \\ h_{m1} \\ \vdots \\ h_{mQ} \end{bmatrix}}_{H_Q} + \underbrace{\begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_Q \\ \vdots \\ e_N \end{bmatrix}}_{E_N}
\tag{5}
$$

Vector H_Q can thus be estimated in the least square sense, in order to minimize $\left(E_N E_N{}^T \right)$ and it is obtained:

$$
H_Q = \left(\Phi_N \Phi_N{}^T \right)^{-1} \Phi_N{}^T Y_N
\tag{6}
$$

However this procedure is quite long according to the value of Q and N and very sensitive to measurement errors.

2.1.2 The correlation technique

A better and faster approach consists in identifying the impulse response $h(t)$, from the cross correlation product of the system response that is the temperature $T_m(t)$ of the sensor and the heat flux $\varphi(t)$ [1]. Indeed, let us rewrite relation (1) taking into account of the measurement errors:

$$
y_m(t) = \int_0^{\infty} h_m(t - \tau) \varphi(\tau) \mathrm{d}\tau + e(t)
\tag{7}
$$

Now let us multiply the two members of this equality by the heat flux $\varphi(t-\tau)$ and integrates from $t=0$ to infinity. We obtain then:

$$\int_0^\infty y_m(t)\varphi(t-\tau)\mathrm{d}t = \int_0^\infty\int_0^\infty h_m(t-\tau)\varphi(\tau)\varphi(t-\tau)\mathrm{d}t\,\mathrm{d}\tau + \int_0^\infty \varphi(t-\tau)e(t)\mathrm{d}t \qquad (8)$$

We see appearing the convolution product between each function as:

$$C_{y_m\varphi}(\tau) = \int_0^\infty h_m(t-\tau)C_{y_m\varphi}(\tau)\mathrm{d}\tau + h_m(t-\tau)C_{e\varphi}(\tau) \qquad (9)$$

If one chose the excitation sequence $\varphi(t)$ as a white noise:

$$C_{\varphi\varphi}(\tau) = \delta(\tau) \qquad (10)$$

And finally, if one admits that the noise measurement is not correlated to the input signal ($C_{e\varphi} = 0$), one has:

$$C_{y_m\varphi}(\tau) = h(\tau) \qquad (11)$$

It thus appears that the impulse response can be directly deducted from the correlation function between the temperature of the sensor and the heat flux. In practice the correlations functions are calculated using he Fast Fourier Transform of the signals (see next section and Matlab code in Appendix 1).

The correlation analysis interest is the physical system identification possibility under less energy constraints density. Indeed in opposition to pulse analysis, the energy does not have to be deposited in an intense way during a very short time (closest to a Dirac function). An interesting feature of such an approach is that the linearity and stationarity assumptions are clearly satisfied and that the confidence domain of the estimated impulse response is the same all over the explored frequency range.

2.1.3 Spectral technique

Nevertheless, this approach is very sensitive to that noise measurement magnitude and practically it the better using the power spectral density instead of the correlation functions [4]:

$$\mathrm{FFT}\left[C_{y_m\varphi}(\tau)\right] = \mathrm{FFT}\left[\int_0^\infty h_m(t-\tau)C_{\varphi\varphi}(\tau)\mathrm{d}\tau\right] = Y_m(f)\Phi(f) = S_{y_m\varphi}(f) \qquad (12)$$

and

$$\mathrm{FFT}\left[C_{\varphi\varphi}(\tau)\right] = \mathrm{FFT}\left[\int_0^\infty \varphi(t-\tau)\varphi(\tau)\mathrm{d}\tau\right] = \Phi(f)^2 = S_{\varphi\varphi}(f) \qquad (13)$$

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$Y_m(f)$ and $\Phi(f)$ are the Fourier transforms of the temperature and the heat flux respectively as well as $S_{\varphi\varphi}(f)$ and $S_{y_m\varphi}(f)$ are the auto and cross PSD. Then, by applying the Fourier transform on relation (9) it is immediately obtained:

$$S_{y_m\varphi}(f) = H(f)S_{\varphi\varphi}(f) + S_{\varphi e}(f) \tag{14}$$

Finally, assuming that the noise measurement is not correlated with the heat flux ($S_{\varphi e}(f) = 0$), the expression of the transfer function is:

$$H(f) = \frac{S_{y_m\varphi}(f)}{S_{\varphi\varphi}(f)} \tag{15}$$

Since the length of the experiment is set to a fixed value τ, the real input signal is:

$$\varphi_\Pi(t) = \varphi(t)\Pi_\tau(t) \tag{16}$$

In this relation, $\Pi_\tau(t) = 1$ when $0 \le t \le \tau$ and 0 elsewhere. Then applying the Fourier transform on the heat flux leads to:

$$\Phi_\Pi(f) = \Phi(f) * \left(\tau \frac{\sin(\pi\tau f)}{\pi\tau f} \right) \tag{17}$$

It appears that the Fourier transform of the heat flux is convoluted by the sinus cardinal function. Usually, the heat flux is pre windowed by a specific function $g_\tau(t)$ which decreases the influence of the function $\Pi_\tau(t)$ as:

$$\varphi_\Pi(t) = \varphi(t)g_\tau(t) \tag{18}$$

For example, it is often used of the Hanning window [3][4] defined by:

$$g_\tau(t) = 0.5\left(1 - \cos\left(\frac{2\pi t}{\tau} \right) \right) \tag{19}$$

It is also used an improved estimation of $S_{y_m\varphi}(f)$ and $S_{\varphi\varphi}(f)$ proposed by Welch [5]. The method consists in dividing the time series data into possible overlapping segments, computing the auto and cross power spectral densities and averaging the estimates.

2.2 The parametric approach

The principles of the system identification method are presented by Ljung [1]. Assuming a linear and stationary system, that means that the thermal properties of the system do not vary with temperature and time, the method consists in identifying the parameters involved in a linear relation between the heat flux $\varphi(t)$ and the temperature $T_m(t)$ of the sensor, from measurements of these two quantities.

Without any kind of physical consideration of the heat transfer process, it is assumed a general relationship of the following form:

$$T_m(t) + \alpha_1 \frac{\mathrm{d}T_m(t)}{\mathrm{d}t} + \alpha_2 \frac{\mathrm{d}^2 T_m(t)}{\mathrm{d}t^2} + \cdots = \beta_0 \varphi(t) + \beta_1 \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} + \beta_2 \frac{\mathrm{d}^2 \varphi(t)}{\mathrm{d}t^2} + \cdots \qquad (20)$$

This kind of model is consistent with the behaviour of the dynamical systems and it is also in case of thermal systems since the heat diffusion equation rests on the first order derivative of the temperature for all the points of the system. It is thus reasonable to admit that the temperature at time t must depend on the heat flux value at time t and also at previous times. On the other hand, since temperature at times before t depend on the heat flux at previous times also, it is not surprising that they appear in the model.

Let us illustrate it on a simple configuration by considering the one dimensional heat transfer in a wall (thermal conductivity k and thermal diffusivity a) subjected to the heat flux density $\varphi(t)$ at $x = 0$ and insulated on the other face at $x = e$. The model thus:

$$\frac{\partial T(x,t)}{\partial t} = a \frac{\partial^2 T(x,t)}{\partial x^2}, \ 0 < x < e, t > 0 \qquad (21)$$

Boundary conditions are:

$$-k \frac{\partial T(x,t)}{\partial x} = \varphi(t), \ x = 0, t > 0 \qquad (22)$$

$$\frac{\partial T(x,t)}{\partial x} = 0, \ x = e, t > 0 \qquad (23)$$

And the initial condition is chosen as:

$$T(x,t) = 0, \ 0 \le x \le e, t = 0 \qquad (24)$$

Let us examine the temperature at $x = e$ and we note $T_m(t) = T(x = e, t)$. Using the Laplace transform $L\{\ \}$ to solve previous problem it is obtained:

$$L\{T_m(t)\} = \theta_m(s) = \frac{1}{k \beta \sinh(\beta e)} L\{\varphi(t)\} = \frac{1}{k \beta \sinh(\beta e)} \Phi(s) \qquad (25)$$

Where: $\beta = \sqrt{s/a}$. The hyperbolic function can be expressed as the following series:

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \ \forall z \ge 0 \qquad (26)$$

Replacing this expression in relation (25) it is found:

$$\theta_m(s) = \cfrac{1}{k\,\beta\,\sum\limits_{n=0}^{\infty}\cfrac{(\beta e)^{2n+1}}{(2n+1)!}}\Phi(s) = \cfrac{1}{k\,\sum\limits_{n=0}^{\infty}\cfrac{e^{2n+1}s^{n+1}}{a^{n+1}(2n+1)!}}\Phi(s) \tag{27}$$

That can be also written as:

$$\sum_{n=0}^{\infty}\alpha_n\, s^{n+1}\,\theta_m(s) = \Phi(s) \tag{28}$$

With: $\alpha_n = k\,\dfrac{e^{2n+1}}{a^{n+1}(2n+1)!}$.

At this stage we must remind us an important property related to the Laplace transform of the derivative of a function:

$$L\left(\frac{d^n f(t)}{dt^n}\right) = s^n\,F(s) - \sum_{k=0}^{n-1}s^{n-k-1}\frac{d^k f(0)}{dt^k} \tag{29}$$

Given to the initial condition (24) it thus appear that relation (28) is equivalent to:

$$\sum_{n=0}^{\infty}\alpha_n\,\frac{d^{n+1}T_m(t)}{dt} = \varphi(t) \tag{30}$$

It is therefore demonstrated that the heat transfer model expressing the temperature at $x = e$ according to the heat flux $\varphi(t)$ imposed at $x = 0$ can be put on the form of the relation (20). In fact the series in (30) can be significantly truncated and we will thus obtain a *low order model*.

Using the discrete form of the derivatives an equivalent form of relation (20) that lead to express the temperature at time $k\,\Delta t$ from the heat flux and the temperature at previous times as:

$$T_m(k) = b_0\,\varphi(k) + b_1\,\varphi(k-1) + b_2\,\varphi(k-2) + \cdots - a_1 T_m(k-1) - a_2 T_m(k-2) - \cdots \tag{31}$$

Let us note that replacing the temperature at previous times with the measurement in relation (31)leads to the predictive model as:

$$T_m(k) = b_0\,\varphi(k) + b_1\,\varphi(k-1) + b_2\,\varphi(k-2) + \cdots - a_1\,y_m(k-1) - a_2\,y_m(k-2) - \cdots \tag{32}$$

Relation (31) is called the output error model whereas relation (32) is called the predictive model. Identification of parameters (a_i, b_j) will significantly differ according to the choice of the model as represented in Figure 6.

Figure 6: parameter identification according to the model representation (output error or predictive).

In case of the output error model configuration, the sensitivity functions $S(a_i) = dT_m(t)/da_i$ and $S(b_j) = dT_m(t)/db_j$ depend on the parameters a_i and b_j. It means that the minimization of $\rho(N) = \sum_{k=0}^{N} \varepsilon(k)^2$ requires non linear minimization algorithm. On the other side, the sensitivity functions do not depend anymore on the parameters when minimizing the quantity $r(N) = \sum_{k=0}^{N} e(k)^2$.

It means that estimation of the parameters in case of the predictive model appeals on a linear minimization algorithm.

2.2.1 Output error model

Let us assume that the number of parameters is n for a_i and $(n+1)$ for b_j. The sensitivity functions of the temperature at time $k\Delta t$ with respect to a_i and b_j are:

$$S_{a_i}(k) = \frac{\partial T_m(k)}{\partial a_i}, \; i = 1,\ldots,n \tag{33}$$

$$S_{b_i}(k) = \frac{\partial T_m(k)}{\partial b_i}, \; i = 0,\ldots,n \tag{34}$$

According to relation (31), it is obtained:

$$S_{a_i}(k) + a_1 S_{a_i}(k-1) + \cdots + a_n S_{a_i}(k-n) = -T_m(k-i), \; i = 1,\ldots,n \tag{35}$$

With: $S_{a_i}(0) = S_{a_i}(1) = \cdots = S_{a_i}(n-1) = 0$

And :

$$b_0 S_{b_i}(k) + b_1 S_{b_i}(k-1) + \cdots + b_n S_{b_i}(k-n) = \varphi(k-i), \; i = 0,\ldots,n \tag{36}$$

With: $S_{b_i}(0) = S_{b_i}(1) = \cdots = S_{b_i}(n-1) = 0$.

Therefore, the output error at time $k\Delta t$ is:

$$\varepsilon(k) = y_m(k) - T_m(k) = \sum_{i=1}^{n} S_{a_i}(k)\Delta a_i + \sum_{i=0}^{n} S_{b_i}(k)\Delta b_i \tag{37}$$

Let us imagine that measurements are collected from $n\Delta t$ up to $N\Delta t$. It is thus obtained a matrix representation of (37) on the form:

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$$\mathbf{E} = \begin{bmatrix} \varepsilon(n) \\ \varepsilon(n+1) \\ \vdots \\ \varepsilon(N) \end{bmatrix} = \mathbf{S} \begin{bmatrix} \Delta a_1 \\ \vdots \\ \Delta a_n \\ \Delta b_0 \\ \vdots \\ \Delta b_n \end{bmatrix} = \mathbf{S} \Delta \Theta \tag{38}$$

Where:

$$\mathbf{S} = \begin{bmatrix} S_{a_1}(n) & \cdots & S_{a_n}(n) & S_{b_0}(n) & \cdots & S_{b_n}(n) \\ \vdots & & \vdots & \vdots & & \vdots \\ S_{a_1}(N) & \cdots & S_{a_n}(N) & S_{b_0}(N) & \cdots & S_{b_n}(N) \end{bmatrix} \tag{39}$$

Solving relation (38) in the least square sense lead to:

$$\Delta \Theta = \left(\mathbf{S}^T \mathbf{S} \right)^{-1} \mathbf{S}^T \mathbf{E} \tag{40}$$

It is thus possible to obtain the optimal value of Θ using an iterative scheme as:

$$\Theta_v = \Theta_{v-1} + \Delta \Theta_{v-1} \tag{41}$$

2.2.2 Predictive model

Relation (32) can be put on the form:

$$y_m(k) = \mathbf{H}(k)\Theta + e(k) \tag{42}$$

Where $\Theta^T = \begin{bmatrix} a_1 & \cdots & a_n & b_0 & \cdots & b_n \end{bmatrix}$ and $\mathbf{H}$ is the regression vector defined as:

$$\mathbf{H}(k) = \begin{bmatrix} -y_m(k-1) & \cdots & -y_m(k-n) & \varphi(k) & \cdots & \varphi(k-n) \end{bmatrix} \tag{43}$$

Let us imagine that measurements are collected from $n\Delta t$ up to $N\Delta t$. Therefore, relation (42) leads to:

$$\mathbf{Y}_N = \mathbf{\Psi}_N \, \Theta + \mathbf{E}_N \tag{44}$$

Where:

$$\mathbf{Y}_N{}^T = \begin{bmatrix} y_m(n) & \cdots & y_m(N+n) \end{bmatrix}, \; \mathbf{\Psi}_N{}^T = \begin{bmatrix} \mathbf{H}(n) & \cdots & \mathbf{H}(N+n) \end{bmatrix} \text{ and } \mathbf{E}_N{}^T = \begin{bmatrix} e(n) & \cdots & e(N+n) \end{bmatrix}$$

It is obtained an estimation of Θ in the linear least square sense as:

$$\Theta = \left(\mathbf{\Psi}_N \mathbf{\Psi}_N{}^T \right)^{-1} \mathbf{\Psi}_N{}^T \mathbf{Y}_N \tag{45}$$

Despite of the rapidity of the method, it must be noted that the estimation is biased. Indeed, let us replace the expression of the identified parameters, relation (45), in the model, relation (42). It is found:

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$$\Theta = \Theta + \left(\Psi_N \Psi_N^{\ T} \right)^{-1} \Psi_N^{\ T} \mathbf{E}_N \tag{46}$$

It is demonstrated in the literature that:

$$E\{\Theta\} = \Theta + \left(E\{\mathbf{H}(k)\mathbf{H}(k)^T\} \right)^{-1} E\{\mathbf{H}(k)^T e(k)\} \tag{47}$$

It thus appears that if $e(k)$ is correlated with $\mathbf{H}(k)$ or if $E\{e(k)\}$ in not zero, the estimation is biased and $E\{\Theta\} \neq \Theta$.

In order to accelerate the identification of Θ, it can be used a recursive scheme. The vector of parameters at instant t is estimated from parameters estimated previously at instant $(t-1)$ according to:

$$\Theta(k) = \Theta(k-1) + \mathbf{L}(k) \left[y_m(k) - \mathbf{H}(k)\Theta(k-1) \right] \tag{48}$$

With:

$$\mathbf{L}(k) = \frac{\mathbf{P}(k-1)\mathbf{H}(k)^T}{\lambda(k) + \mathbf{H}(k)\mathbf{P}(k-1)\mathbf{H}(k)^T}$$

And:

$$\mathbf{P}(k) = \mathbf{P}(k-1) - \frac{\mathbf{P}(k-1)\mathbf{H}(k)^T \mathbf{H}(k)\mathbf{P}(k-1)}{\lambda(k) + \mathbf{H}(k)\mathbf{P}(k-1)\mathbf{H}(k)^T}$$

where the initial values are: $\Theta(0) = \mathbf{0}_D$ and $\mathbf{P}(0) = 10^6 \mathbf{I}_D$, with $\mathbf{0}_D$ and $\mathbf{I}_D$ are zeros vector and ones matrix respectively with dimension $D = 2N$.

Remark: unbiased approaches are proposed in the literature that consist in whitened the sequence $e(k)$ in relation (42). This is the instrumental variables method, and methods based on the change of the model structure (auto regressive with exogene input model, auto regressive with adjusted mean and exogene input model for example).

3 Application

Let us consider the heat transfer problem presented above and let us generate a heat flux sequence on the form of the pseudo random binary sequence represented in Figure 7. The choice of such a sequence for the excitation is that it is quite easy to make in practice and it is also very close to a white noise in terms of the power spectral density as represented in Figure 8.
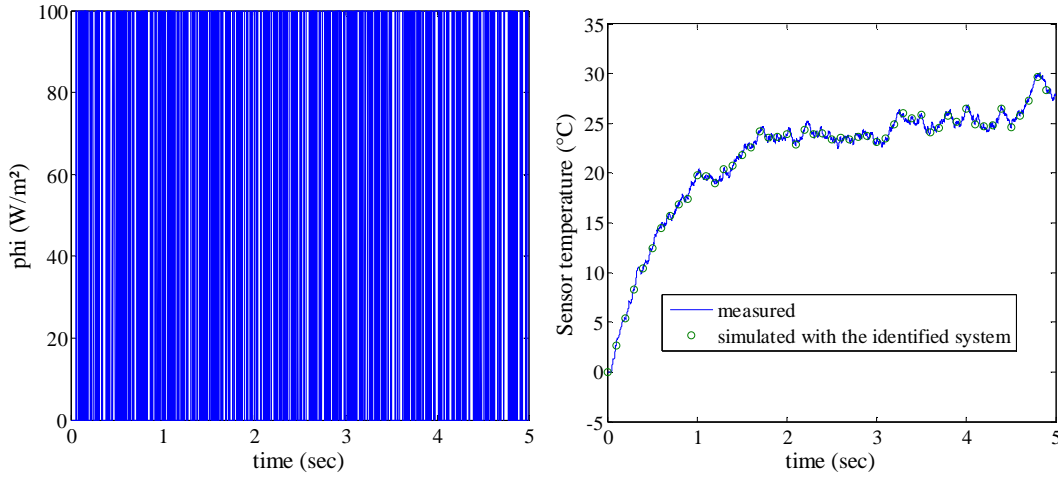
Figure 7: image on the left – heat flux generated on the form of a PRBS; image on the right – measured temperature of the sensor and comparison with the simulation of the identified system.
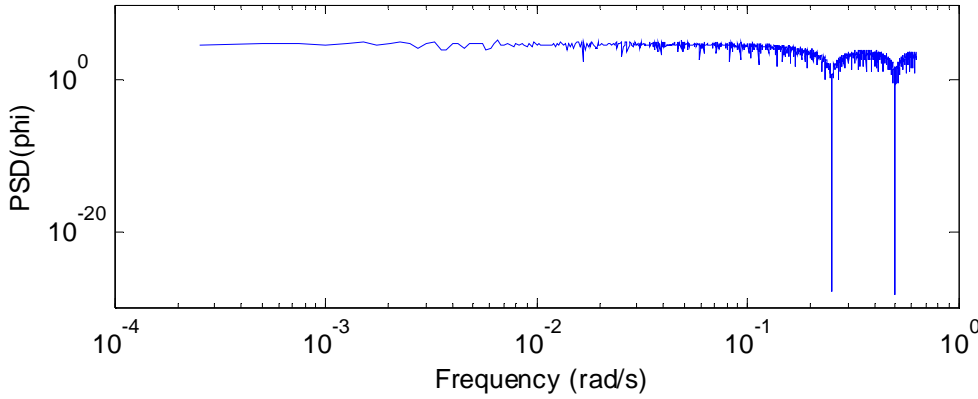

Figure 8: power spectral density of the heat flux generated as a PRBS.

Using the correlation method described previously, it is obtained the impulse response represented in Figure 9. As viewed on this figure, the impulse response reconstructed using the correlation technique is very sensitive to noise measurement.

In a second stage, we used the parametric approach in order to find the model on the form of the relation (32) that fits the experimental measurements (Figure 7) at the best. The choice of $\mathbf{\Lambda} = [na, nb]$ (*na* is the number of parameters a_i and *nb* is the number of parameters b_i) is made by collecting in a matrix all the values of $\mathbf{\Lambda}$ to be investigated and looking on the value of the Aikake [1] criterion defined by

$$\Psi = \frac{1 + n/N}{1 - n/N} V \, , \ n = na + nb + 1 \tag{49}$$

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where n is the total number of estimated parameters and V is the loss function defined by

$$V = \sum_{k=1}^{N} e_k^{\,2}$$

(50)

Standard errors of the estimates are calculated from the covariance matrix of Θ. If the assumptions of additive, zeros mean, constant variance σ^2 and uncorrelated errors are verified, the covariance matrix is expressed as

$$\text{cov}(\Theta) = (\mathbf{H}^T\,\mathbf{H})^{-1}\,\sigma^2$$

(51)

An estimate of the variance σ^2, denoted s^2, is:

$$s^2 = \frac{1}{N-n}\mathbf{E}^T\,\mathbf{E}$$

(52)

It is found the optimal set of parameters (a_i, b_i) as:

Parameter	value	Standard deviation	Parameter	value	Standard deviation
a_0	1	0	a_5	0.0166	0.0054
a_1	0.2823	0.01364	b_0	0.0007006	5.348e-006
a_2	0.2539	0.01368	b_1	0.0006788	1.19e-005
a_3	0.2715	0.01375	b_2	0.0004693	1.404e-005
a_4	0.2047	0.01427	b_3	0.0002561	1.365e-005

The loss function is V=0.000123859.

Simulating the response with the heat flux sequence it is obtained a very good agreement with measured data as represented in Figure 7. Therefore it is simulated the impulse response from the identified system and it is reported the results in Figure 9. It is found a very nice agreement with that calculated from the FEM. The main difference occurs at the short time.
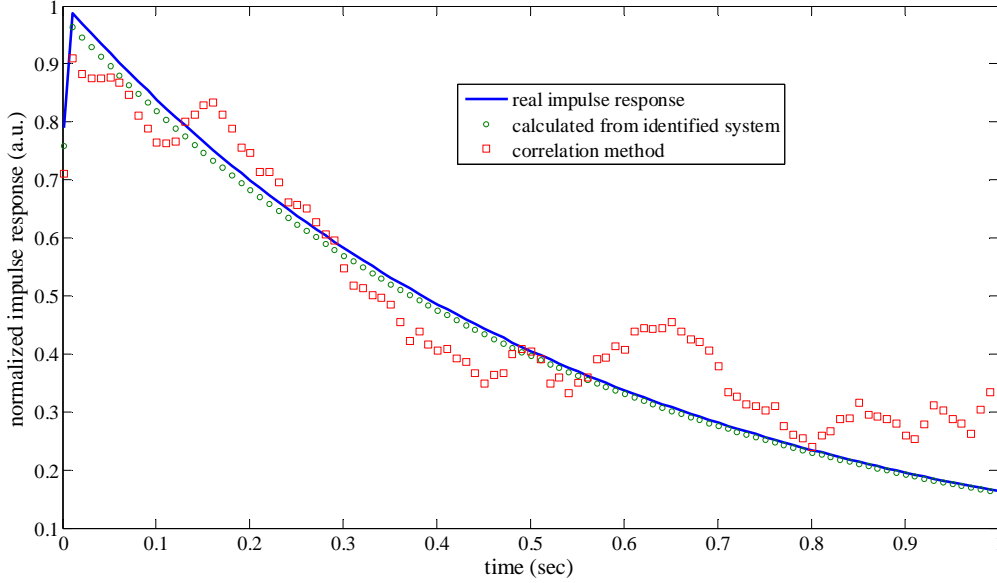
Figure 9: real impulse response and impulse response found using the correlation method and the parametric method.

4 Let's go a little further

Let us consider again the configuration of heat transfer in a wall studied before but let us focus now on the temperature $T_m(t)$ at $x = 0$ where the heat flux is applied. Using the Laplace transform to solve the heat diffusion equation with associated boundary and initial conditions (relations (21) to (24)), it is obtained [10]:

$$L\{T_m(t)\} = \theta_m(s) = \frac{\cosh(\beta e)}{k\beta \sinh(\beta e)} L\{\varphi(t)\} = \frac{\cosh(\beta e)}{k\beta \sinh(\beta e)} \Phi(s) \tag{53}$$

Where: $\beta = \sqrt{s/a}$. The hyperbolic functions can be expressed as the following series:

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \quad \text{and} \quad \sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \quad \forall z \tag{54}$$

Replacing these expressions in relation (53) it is found:

$$\theta_m(s) = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{(\beta e)^{2n}}{(2n)!}}{k\beta \displaystyle\sum_{n=0}^{\infty} \frac{(\beta e)^{2n+1}}{(2n+1)!}} \Phi(s) = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{e^{2n} s^n}{a^n (2n)!}}{k \displaystyle\sum_{n=0}^{\infty} \frac{e^{2n+1} s^{n+1}}{a^{n+1} (2n+1)!}} \Phi(s) \tag{55}$$

That can be also written as:

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$$\sum_{n=0}^{\infty} \alpha_n \, s^{n+1} \, \theta_m(s) = \sum_{n=0}^{\infty} \beta_n \, s^n \, \Phi(s) \tag{56}$$

With: $\alpha_n = k \dfrac{e^{2n+1}}{a^{n+1}(2n+1)!}$ and $\beta_n = \dfrac{e^{2n}}{a^n(2n)!}$.

Given to the initial condition (24) and using the property (29) it thus appear that relation (56) is equivalent to:

$$\sum_{n=0}^{\infty} \alpha_n \frac{\mathrm{d}^{n+1}T_m(t)}{\mathrm{d}t} = \sum_{n=0}^{\infty} \beta_n \frac{\mathrm{d}^n \varphi(t)}{\mathrm{d}t} \tag{57}$$

It is therefore demonstrated that the heat transfer model expressing the temperature at $x=0$ according to the heat flux $\varphi(t)$ imposed at $x=0$ can be put on the form of the relation (20). However, if one tries to fit experimental data by simulating the model in relation (57) it appears that it is necessary to keep a very important number of terms in the series in order to reproduce accurately the transient response at the short times. In that case relation (57) cannot be viewed as a lower order model and moreover, the identification of parameters $\{\alpha_n, \beta_n\}$ becomes inaccurate when n becomes large. It means that the model structure on the form of the relation (20) is not optimal for all the possible configurations.

Let's try first to understand such an observation and let's try to find a better low order model structure that would approach the searched optimality.

The raison why model (20) is not available for describing the behaviour at the short times is given in the expression of the asymptotic behaviour at the short times. Indeed, relation (53) shows that:

$$\lim_{s \to \infty} \frac{\cosh\left(e\sqrt{s}/\sqrt{a}\right)}{k\,e\sqrt{s}/\sqrt{a}\,\sinh\left(e\sqrt{s}/\sqrt{a}\right)} = \frac{1}{k/\sqrt{a}\,\sqrt{s}} \tag{58}$$

On the other hand taking the same limit for relation (55) give:

$$\lim_{s \to \infty} \frac{\sum_{n=0}^{\infty} \beta_n \, s^n}{\sum_{n=0}^{\infty} \alpha_n \, s^{n+1}} = \frac{\beta_n \, s^n}{\alpha_n \, s^{n+1}} = \frac{2n+1}{e\,k/a} \frac{1}{s} \tag{59}$$

It is thus obvious that relation (53) and equivalent relation (55) do not have the same asymptotic behaviour at the short times. In other words the exact asymptotic behaviour, described by relation (58), is that of the semi infinite medium ($\propto 1/\sqrt{s}$) whereas that of the equivalent model describes a capacitance effect ($\propto 1/s$). It means that the contribution of an infinite number of derivatives is theoretically required to approach the semi infinite behaviour of the system.

It is then possible to find a better low order model that will respect the asymptotic behaviour at the short times? The answer is fortunately yes thanks to the works of Liouville in the 19th century [11][12]. He demonstrated that the property:

$$L\left(\frac{d^{\nu} f(t)}{dt^{\nu}}\right) = s^{\nu} F(s) - \sum_{k=0}^{n-1} s^{n-k-1} \frac{d^{\nu} f(0)}{dt^{\nu}}$$

(60)

Remains exact even if ν is real and more generally complex. $D^{\nu}\{f(t)\} = d^{\nu} f(t)/dt^{\nu}$ is called the derivative of real order ν (often called the non integer derivative in order to discriminate from the classical derivative) and is defined as [13][14][15]:

$$D^{\nu}\{f(t)\} = D^{n}\left\{I^{n-\nu}\{f(t)\}\right\} \qquad n \in N, \operatorname{Re}(\nu) > 0, \quad n-1 \leq \operatorname{Re}(\nu) < n$$

(61)

where the integral or real order ν is defined in the Liouville sense as:

$$I^{\nu}\{f(t)\} = \frac{1}{\Gamma(\nu)} \int_{0}^{t} (t-u)^{\nu-1} f(u) du, \quad \operatorname{Re} \nu > 0$$

(62)

With:

$$\Gamma(\nu) = \int_{0}^{\infty} u^{\nu-1} \exp(-u) du$$

(63)

Regarding to relation (58), it is now clear that:

$$L^{-1}\left\{\frac{1}{k/\sqrt{a}} \frac{1}{\sqrt{s}} \Phi(s)\right\} = \frac{1}{k/\sqrt{a}} I^{1/2}\{\varphi(t)\}$$

(64)

Finally, we can assert that, instead of relation (20), an optimal structure of a low order model for heat transfer problem by diffusion must be of the following form:

$$\sum_{n=0}^{\infty} \alpha_{n} D^{n/2}\{T_{m}(t)\} = \sum_{n=0}^{\infty} \beta_{n} D^{n/2}\{\varphi(t)\}$$

(65)

Let us demonstrate it on the 1D heat diffusion problem in a wall when $T_{m}(t)$ is the temperature at $x = 0$. We saw that we could not find an equivalence of the exact solution (53) on the form of relation (65). In fact it comes from the manner we have replaced the hyperbolic functions with their series. Let us use the expression of the hyperbolic functions from the exponential:

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2} = \frac{e^{-z}(1 + e^{2z})}{2}$$

(66)

And:

$$\sinh(z) = \frac{e^z - e^{-z}}{2} = \frac{e^{-z}\left(-1 + e^{2z}\right)}{2} \tag{67}$$

Replacing these expressions in relation (53) give:

$$\theta_m(s) = \frac{e^{2\beta e}}{k\,\beta\left(e^{2\beta e} - 1\right)}\Phi(s) \tag{68}$$

The series of the exponential is:

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}, \quad \forall z \tag{69}$$

Replacing this decomposition in relation (68) lead to:

$$\theta_m(s) = \frac{\displaystyle\sum_{n=0}^{\infty}\beta'_n s^{n/2}}{\displaystyle\sum_{n=0}^{\infty}\alpha'_n s^{(n+1)/2}}\Phi(s) \tag{70}$$

With:

$$\alpha'_n = \frac{k\,e^{n-1}}{a^{(n-1)/2}\,n!} \qquad and \qquad \beta'_n = \frac{e^{n-1}}{a^{(n-2)/2}\,n!} \tag{71}$$

It is now well finding a consistent equivalent expression of the exact solution whose asymptotic behaviour at the short times ($z \to \infty$) is exactly the relation (58). Going back to time, relation (70) becomes [16][17]:

$$\sum_{n=0}^{\infty}\alpha'_n \mathrm{D}^{(n+1)/2}\left\{T_m(t)\right\} = \sum_{n=0}^{\infty}\beta'_n \mathrm{D}^{n/2}\left\{\varphi(t)\right\} \tag{72}$$

Let us insist on the fact that relations (57) and (72) are both exact. The difference lies in the fact that an infinite number of terms are required in relation (57) to describe the response at the short times, when the system behaves as a semi infinite medium, whereas only one is necessary using relation (72). The Matlab code for the implementation of the technique is given in Appendix 1. It is used the recursive approach presented previously for the classical (with integer derivatives) parametric technique.

5 Conclusion

System identification is a powerful tool that allows the user to obtain a direct model to solve an inverse problem. In fact, this approach will consist in applying a known thermal excitation and to measure the temperature at the sensors in order to find a relationship between these two quantities. Obviously, this approach find an interest if the system is not well characterized in terms of its thermal properties (thermal conductivity, specific heat, density, heat exchange coefficient at the boundaries,

thermal resistance at the interfaces). Moreover, this technique does not require knowing the exact locations of the sensors in the system as well as their dynamical behaviour. It means that it is not required making a calibration of the sensors since they are used both for the system identification and the inversion. The constraints encountered with such an approach are that the system must be identified in the same configuration in which it will be during the inversion. It means first that the time range for the system identification will define the time domain of use for the direct model. On the other hand, all of the boundary conditions experienced during the system identification must remain identical during the inversion.

Finally, it must be emphasized than the computational times for the inversion will be decreased very significantly even if the thermal system is complex. It is a very interesting feature of this approach since the simulation of the identified system is faster than that based on a discretization of the heat equation.

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7 Appendix 1: Matlab codes

We denote z=[y u], the experimental data, where u is the input (heat flux) and y is the output (the temperature of the sensor)

7.1 Correlation method

```
function ir=correlation(z,M)
%     ir: the estimated impulse response
%     M: The number of lags for which the functions are computed

Rft = covar(z,M+1);
r(:,1) = (-M:1:M)';
```

```
r(M+1:2*M+1,2:3) = Rft([1 4],:)';
r(1:M,2:3) = Rft([1 4],M+1:-1:2)';
scir = Rft(4,1); sccf = sqrt(Rft(1,1)*Rft(4,1));
r(M+1:2*M+1,4) = Rft(2,:)'/sccf;
r(1:M,4) = Rft(3,M+1:-1:2)'/sccf;
ir = r(M+1:2*M+1,4)*sccf/scir;


function R=covar(z,M)
% Computes the covariance for z
%   M: The maximum delay - 1, for which the covariance function is
estimated.

[Nft,nz]=size(z);
nfft = 2.^ceil(log(2*Nft)/log(2));
Yft=fft(([z(:,1)' zeros(1,Nft)]),nfft);
Uft=fft(([z(:,2)'  zeros(1,Nft)]),nfft);
YUft=Yft.*conj(Uft);
UAft=abs(Uft).^2;
YAft=abs(Yft).^2;
RYft=fft(YAft,nfft);
n=length(RYft);
sumnft = sumnft+Nft;
R=real(RYft(1:M))/n;
```

7.2 Spectral method

```
function H = TF(z,N,M)

% The transfer function H is estimated at N equally spaced frequencies
between 0 (excluded) and pi.
% A smoothing operation is performed on the raw spectral estimates using a
Hamming Window, giving a frequency resolution of about pi/M.


[Ncap,nz] = size(z);
M = M/2; % this is to make better agreement with SPA.
M1 = fix(l/M);sc=l/(2*N);
u = z(:,2);
y = z(:,1);
nfft = 2*ceil(Ncap/N)*N;
Yft = fft(y,nfft,1);
Uft = fft(u,nfft,1);
Yft = [Yft(l-M1+2:l,:);Yft];
Uft = [Uft(l-M1+2:l,:);Uft];
Yft = Yft.*conj(Uft);
Uft = abs(Uft).^2;
ha = .54 - .46*cos(2*pi*(0:M1)'/M1);
ha = ha/(norm(ha)^2);
Yft = filter(ha,1,Y);
Uft = filter(ha,1,U);
Yd = Yd+Yft(M1+fix(M1/2)+sc:sc:M1+fix(M1/2)+l/2,:,:);
Ud = Ud+Uft(M1+fix(M1/2)+sc:sc:l/2+M1+fix(M1/2),:);
H = Yd./Ud;
```

7.2.1   Parametric estimation

```
function [n_ord,num,d_ord,den,rsdi,ecn,ecd] =
ni_sid_ident_rec(u,y,time,num_def,den_def,adm,adg,teta0)
```

```matlab
%
% Fonction ni_sid_ident_rec
%
% Identification of non integer model using recusive least square algorithm
%
% Input Argument
%
%   u,y: system input and output
%   time: time vector
%   num_def: numerator (first line orders and second line 0 for unknown
%   parameters else give the value)
%   den_def: denominator (first line orders and second line 0 for unknown
%   parameters else give the value)
%   adm: Adaptation mechanism. adg: Adaptation gain
%       adm='ff', adg=lam:  Forgetting factor algorithm, with forg factor
lam
%       adm='kf', adg=R1: The Kalman filter algorithm with R1 as covariance
%           matrix of the parameter changes per time step
%       adm='ng', adg=gam: A normalized gradient algorithm, with gain gam
%       adm='ug', adg=gam: An Unnormalized gradient algorithm with gain gam
%   teta0: initial value of the parameters
%
% Output Argument
%
%   num,den: denominator and numerator coefficent
%   n_ord, d_ord: order of the numerator and denominator
%   rsdi: residuals
%   ecn, ecn: standard deviation for the estimated parameters
%
% Jean-Luc Battaglia
%
adm=lower(adm(1:2));
if ~(adm=='ff'|adm=='kf'|adm=='ng'|adm=='ug')
 error('The argument ADM should be one of ''ff'', ''kf'', ''ng'', or
''ug''.')
end
if adm(2)=='g', grad=1;else grad=0;end
%
n_ord=num_def(1,:); d_ord=den_def(1,:);                         %
derivation order
d_ord_ukn=find(den_def(2,:)==0); n_ord_ukn=find(num_def(2,:)==0);   %
orders associated to unnkown parameters
d=length(d_ord_ukn)+length(n_ord_ukn);                         %
number of unknown parameters
d_ord_knw=find(den_def(2,:)~=0); n_ord_knw=find(num_def(2,:)~=0);   %
orders associated to unnkown parameters
%
p=10000*eye(d);
if nargin < 8, teta=eps*ones(d,1); else teta=teta0; end
if adm(1)=='f', R1=zeros(d,d);lam=adg;end;
if adm(1)=='k', [sR1,SR1]=size(adg);
    if sR1~=d | SR1~=d
        error(['The R1 matrix should be a square matrix with dimension
',...
            'equal to number of parameters.'])
    end;
    R1=adg;lam=1;
end;
%
Yf=dn(time(2)-time(1),y,d_ord); Uf=dn(time(2)-time(1),u,n_ord); % matrice
de régression complète (pour tous les ordres)
```

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```matlab
%
phi=[-Yf(:,d_ord_ukn) Uf(:,n_ord_ukn)]; % regression vector
%
yn=[Yf(:,d_ord_knw) -Uf(:,n_ord_knw)]*[den_def(2,d_ord_knw)
num_def(2,n_ord_knw)]';
%
rsdi=0;
for kcou = 1:length(u)-1,
    yh=phi(kcou,:)*teta;          % ym(t+1)
    if ~grad,K=p*phi(kcou,:)'/(lam + phi(kcou,:)*p*phi(kcou,:)');% k(t+1)
        p=(p-K*phi(kcou,:)*p)/lam+R1;    % p(t+1)
    else K=adg*phi(kcou,:);end;
    if adm(1)=='n', K=K/(eps+phi(kcou,:)*phi(kcou,:)');end;
    epsi=yn(kcou)-yh;            % y(t+1)-ym(T+1)
    rsdi=rsdi+epsi^2;
    teta=teta+K*epsi;          % pmc(t+1)=pmc(t)+k(t+1)*(y(t+1)-ym(T+1))
end;
rsdi=sqrt(rsdi/kcou);
ec_teta=(rsdi/2).*sqrt(diag(p));
%
% Transfert function parameter computation from teta vector
den(d_ord_knw)=den_def(2,d_ord_knw);
den(d_ord_ukn)=teta(1:length(d_ord_ukn));
ecd(d_ord_knw)=0;ecd(d_ord_ukn)=ec_teta(1:length(d_ord_ukn));
num(n_ord_knw)=num_def(2,n_ord_knw);
num(n_ord_ukn)=teta(length(d_ord_ukn)+1:end);
ecn(n_ord_knw)=0;ecn(n_ord_ukn)=ec_teta(length(d_ord_ukn)+1:end);


function [dy,Erreur]=dn(time,x,n)

% [dy,Erreur]=dn(time,x,n)
%
% This function computes the derivate of order n, with n complex vector,
% of the data x ; time is the sampling period or the time vector
%
% Argument in :
%   time : vector time of the vector x (scalar vector) or sample (scalar)
%   x : data (complex matrix)
%   n : order of the derivate (complex vector)
%
% Argument out :
%   dy : data (complex matrix)
%
%
S_time=size(time);
S_x=size(x);
S_n=size(n);

%sampling time interval
h=[time(2);time(2:end)]-[time(1);time(1:end-1)];
Ak=binome(n,S_x(1));
A=Ak;

%derivative computation
dy=zeros(S_x(1),S_x(2));
y=zeros(S_x(1),1);

for col=1:S_x(2)
    y=conv(x(:,col),A(:,col));
    dy(:,col)=y(1:S_x(1))./h.^n(1,col);
```

```
end;
```

8   Appendix 2: the non integer calculus

Let us consider $f(t)$ an integrable function integrable, definite and bounded, on (a, ∞) upon which we make *n* successive integrations. One obtains:

$$_a\mathrm{I}_t^n \; f(t) = \int_a^t \mathrm{d}u_1 \int_a^{u_1} \mathrm{d}u_2 \cdots \int_a^{u_{n-1}} f(u_n) \; \mathrm{d}u_n = \frac{1}{(n-1)!} \int_a^t (t-u)^{n-1} \; f(u) \; \mathrm{d}u \tag{1}$$

Since $(n-1)! = \Gamma(n)$, it is easier to generalize the previous relation to any number *n* real, and more generally complex, and then to define the integral of real order ν ($\mathrm{Re}\,\nu > 0$), or more simply the non integer integral as:

$$_a\mathrm{I}_t^\nu \; f(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-u)^{\nu-1} \; f(u)\,du \,, \quad \mathrm{Re}\,\nu > 0 \tag{2}$$

With $\Gamma(\nu)$ the Eulerian function of second specie defined by:

$$\Gamma(\nu) = \int_0^\infty u^{\nu-1} \exp(-u)\,du \tag{3}$$

The non integer integral is similar to the convolution product between function $t^{\nu-1}$ and function $f(t)$. It is usual to restrain the lower bound of the integral to $a = 0$, that corresponds to the initial time of the experiment. This leads to the definition of the non integer integral of order ν in the sense of Reimann-Liouville and we note $\mathrm{I}^\nu \; f(t) =_0\mathrm{I}_t^\nu \; f(t)$. The additive property upon the integration order is expressed as:

$$\mathrm{I}^\nu \mathrm{I}^\mu f(t) = \mathrm{I}^{\nu+\mu} f(t) \,, \quad \forall \;\; \mathrm{Re}(\nu,\mu) > 0 \tag{4}$$

This leads to the non integer derivative of order ν as:

$$\mathrm{D}^\nu f(t) = \mathrm{D}^n \; \mathrm{I}^{n-\nu} f(t) \qquad n \in \mathrm{N},\, \mathrm{Re}(\nu) > 0,\;\; n-1 \le \mathrm{Re}(\nu) < n \tag{5}$$

From those definitions, it appears that the non integer derivation of function $f(t)$ at time *t* is expressed according to the entire set of values of the function from the initial time until time *t*. This operator has therefore an infinite memory effect that distinguishes it fundamentally from the classical derivative of integer order. However, the values of the function previous to time *t* are weighted by a fforgotten factor that is as high as one approches the initial time.

The discrete representation of the non integer derivative has been given by Grünwald and is expresses as:

$$\mathrm{D}^\nu f(t) = \lim_{h \to 0} \frac{\Delta_h^\nu \; f(t)}{h^\nu}, \; \nu > 0 \,, \tag{6}$$

Δ_h^ν represents the non integer increase defined by:

$$\Delta_h^v \, f(t) = \sum_{j=0}^{N} (-1)^j \binom{v}{j} f(t - j\,h), \ \ t = N\,h \tag{7}$$

With :

$$\binom{v}{j} = \frac{v(v-1)\cdots(v-j+1)}{j!} \tag{8}$$

Let us note that the sampling time interval *h* must be necessary constant with this definition.