L4 – Non Linear Estimation Problems

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OUTLINE

I. Definition and Vocabulary
II. Useful Tools to investigate NLPE Problems
III. Enhancing the Performances of Estimation
- Natural Parameters & Dimensional Analysis
- Reducing the PEP to make it well- conditioned (Case of the Contrast Method)
- Over-Parameterized Models (Case of the “Hot”-Wire technique
- Estimations with models without degrees of Freedom (Case of the Liquid Flash Experiment)
- Taking the bias into account to reduce the variances on estimated parameters (Case of the classical “Flash” method)

Some Definitions and Vocabulary Precisions

Measuring a physical quantity β_j requires a specific experiment allowing for this quantity to "express itself as much as possible" (notion of **sensitivity**).

This experiment is requires a **system** onto which **inputs** $u(t)$ are applied (stimuli) and whose **outputs** $y(t)$ are collected (observations). t is the **explanatory** variable : it corresponds to time for a pure dynamical experiment.

A model M is required to mathematically express the dependence of the system's response with respect to quantity β_j and to other additional parameters

$$\beta_k \ \left(k \neq j\right) \ : \ \boldsymbol{y}_{mo} = \eta(\boldsymbol{t},\boldsymbol{\beta},\boldsymbol{u})$$

Many candidates may exist for function η -depending on the degree of complexity reached for modelling the physical process- which may exhibit different mathematical structure –depending for example on the type of method used to solve the model equations.

Once this model is established, the physical quantities in vector $\boldsymbol{\beta}$ acquire the status of **model parameters**.

This model (called **knowledge** model if it is derived from physical laws and/or conservation principles) is initially established in a direct formulation.

Knowing inputs $u(t)$ and the value taken by parameter $\boldsymbol{\beta}$, the output(s) can be predicted.

The **linear** or **non linear** character of the model has to be determined:

- A Linear model with respect to its Inputs (LI structure) is such as:

$$y_{mo}(t,\beta,\alpha_1 u_1+\alpha_2 u_2)=\alpha_1 y_{mo}(t,\beta,u_1)+\alpha_2 y_{mo}(t,\beta,u_2) \qquad (1)$$

- A Linear model with respect to its parameters (LP structure) is such as:

$$y_{mo}(t,\alpha_1\beta_1+\alpha_2\beta_2,u)=\alpha_1 y_{mo}(t,\beta_1,u)+\alpha_2 y_{mo}(t,\beta_2,u) \qquad (2)$$

The inverse problem consists in making the direct problem work backwards with the objective of getting (extracting) β from $y_{mo}(t,\beta,u)$ for given inputs and observations $\boldsymbol{y}$. This is an identification process.

The difficulty stems here from two points:

(i) Measurements $\boldsymbol{y}$ are subjected to random perturbations (intrinsic noise $\boldsymbol{\varepsilon}$) which in turn will generate perturbed estimated values $\hat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$, even if the model is perfect: this constitutes an estimation problem.

(ii) the mathematical model may not correspond exactly to the reality of the experiment. Measuring the value of $\boldsymbol{\beta}$ in such a condition leads to a biased estimation $Bias=E(\hat{\beta})-\beta^{true}$: this corresponds to an identification problem (which model η to use ?) associated to an estimation problem (how to estimate β for a given model?).

The estimation/identification process basically tends to make the model match the data (or the contrary). This is made by using some mathematical "machinery" aiming at reducing some gap (distance or norm)

$$\boldsymbol{r}(\boldsymbol{\beta}) = \boldsymbol{y} - \boldsymbol{y}_{mo}(\boldsymbol{t}, \boldsymbol{\beta}, \boldsymbol{u}) \qquad (3)$$

One of the obvious goal of NLPE studies is then to be able to assess the performed estimation through the production of numerical values for the variances $V(\hat{\boldsymbol{\beta}})$ obtained on the estimators (set of estimated values parameter. This allows to give the order of magnitude of confidence bounds for the estimate). NLPE problems require the use of Non Linear statistics for studying such properties of the estimates.

Because of the two above-mentioned drawbacks of MBM, the estimated or measured value of a parameter β_j will be considered as "good" if it is not biased and if its variance is minimum.

Quantifying the bias and variance is also helpful to determine which one of two rival experiments is the most appropriate for measuring the searched parameter (Optimal design). In case of multiple parameters (vector $\boldsymbol{\beta}$) and NLPE problems, it is also helpful to determine which components of vector $\boldsymbol{\beta}$ are correctly estimated in a given experiment.

Useful Tools to Investigate NLPE Problems

Sensitivities

In the case of a single output signal $\boldsymbol{y}$ with m sampling points for the explanatory variable t and for a model involving n parameters, the sensitivity matrix is $(m \times n)$ defined as

$$S_{ij} = \left. \frac{\partial y_{mo}(t_i ; \boldsymbol{\beta}^{nom})}{\partial \beta_j} \right|_{\boldsymbol{t}, \beta_k \text{ pour } k \neq j}$$

As the problem is NL, the sensitivity matrix has only a local meaning. It is calculated for a given nominal parameter vector $\boldsymbol{\beta}^{nom}$.

If the model has a LP structure, this means that the sensitivity matrix is independent from β. It can be expressed as (Lecture 2)

$$y_{mo}(t, \boldsymbol{\beta}) = \sum_{j=1}^{n} S_j(t) \beta_j$$

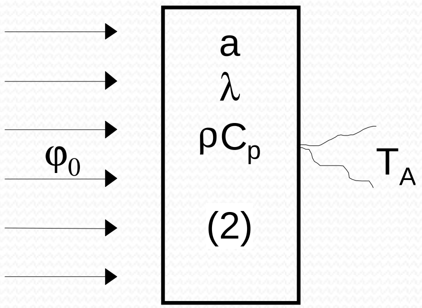
The sensitivity coefficient $S_j(t)$ to the j^{th} parameter β_j corresponds to the j^{th} column of matrix $\boldsymbol{S}$.

The primary way of getting information about the identifiability of the different parameters is to analyse sensitivity the coefficients through graphical observations. This is possible only when considering reduced sensitivity coefficients $\boldsymbol{S}_j^*$ because the parameters of a model do not have in general the same units.

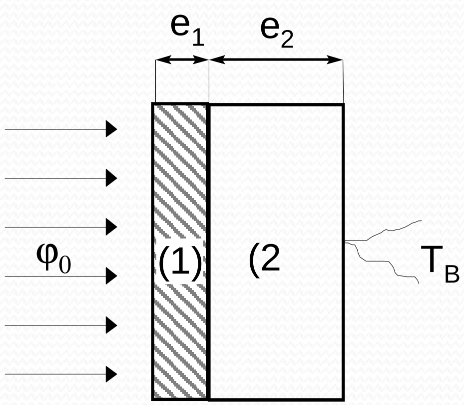
$$\boldsymbol{S}_j^* = \beta_j \boldsymbol{S}_j = \beta_j \left. \frac{\partial \boldsymbol{y}_{mo}(\boldsymbol{t}; \boldsymbol{\beta}^{nom})}{\partial \beta_j} \right|_{\boldsymbol{t}, \beta_k \text{ pour } k \neq j} = \left. \frac{\partial \boldsymbol{y}_{mo}(\boldsymbol{t}; \boldsymbol{\beta}^{nom})}{\partial (\ln \beta_j)} \right|_{\boldsymbol{t}, \beta_k \text{ pour } k \neq j}$$

TOOL Nr1: A superimposed plot of reduced sensitivity coefficients $S_j^*(t)$ gives a first idea about the more influent parameters of a problem (largest magnitude) and about possible correlations (sensitivity coefficients following the same evolution).

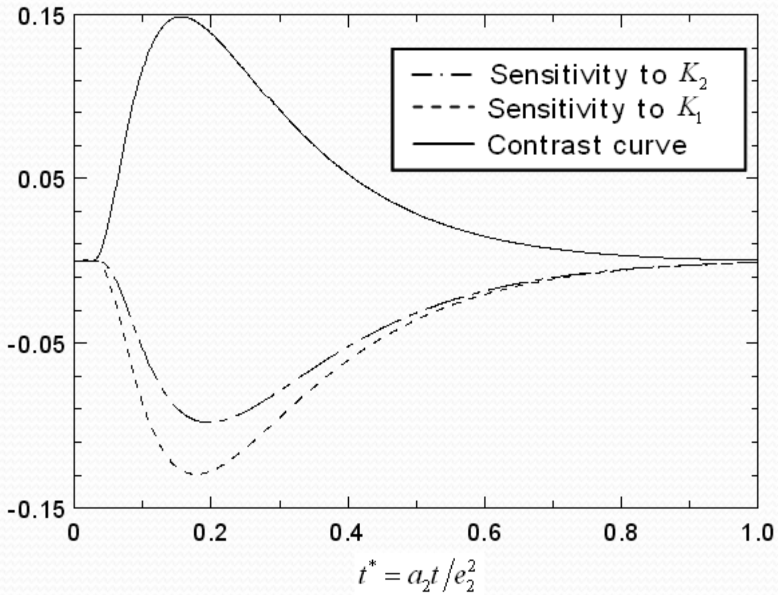
Example: Measurement of thermophysical properties of coatings through Flash method using thermal contrast principle. Case $n = 2$



Experiment **A**



experiment **B**



$$K_1 = \frac{e_1}{e_2}\sqrt{\frac{a_2}{a_1}} \text{ and } K_2 = \sqrt{\frac{\lambda_1 \rho_1 c_1}{\lambda_2 \rho_2 c_2}}$$

Reduced sensitivity coefficients for $K_1 = 0.1$ ***and*** $K_2 = 1.36$

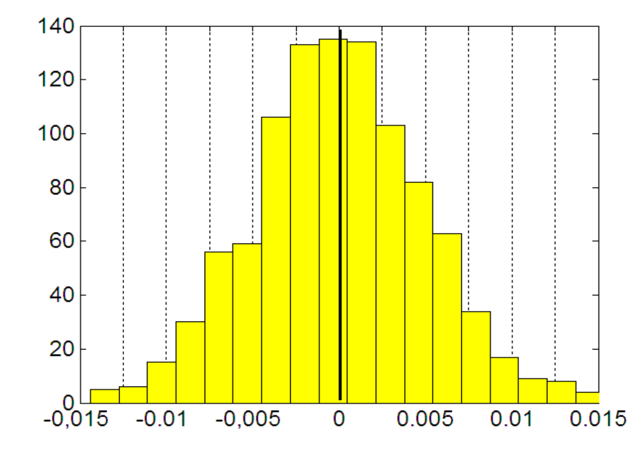
INVERSE ANALYSIS :

The model :

$$T(t_i, \boldsymbol{\beta})$$

The Observable:

$$Y_i = T(t_i, \boldsymbol{\beta}) + \varepsilon_i$$



$$\sigma_{\varepsilon_i} = 0.005$$

The experimental noise corrupt the data:

$$E(\varepsilon_i) = 0 \qquad \mathrm{var}(\varepsilon_i) = \sigma^2 \qquad \mathbf{cov}(\varepsilon_i) = \sigma^2 \mathbf{Id}$$

$$S(\boldsymbol{\beta}) = \sum_{i=1}^{n} \left(Y_i - T(t_i, \boldsymbol{\beta}) \right)^2$$

allows to get an estimate of β via minimization

INVERSE ANALYSIS : The minimization process

$$\hat{\boldsymbol{\beta}}^{(k+1)} = \hat{\boldsymbol{\beta}}^{(k)} + \left(\mathbf{X}^{(k)^t} \mathbf{X}^{(k)} \right)^{-1} \mathbf{X}^{(k)^t} \left(\mathbf{Y} - \mathbf{T}(\boldsymbol{\beta}^{(k)}) \right)$$

indicates the basic tools for inverse analysis

= Matrix analysis

Sensitivities to parameters

$$\mathbf{X} = \nabla_{\boldsymbol{\beta}} \mathbf{T}^{\mathrm{T}}(t, \boldsymbol{\beta}) = \frac{\partial \mathbf{T}(t, \boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \begin{bmatrix} \frac{\partial T(t_1, \boldsymbol{\beta})}{\partial \beta_1} & \frac{\partial T(t_2, \boldsymbol{\beta})}{\partial \beta_1} & \cdots & \frac{\partial T(t_n, \boldsymbol{\beta})}{\partial \beta_1} \\ \vdots & \vdots & & \vdots \\ \frac{\partial T(t_1, \boldsymbol{\beta})}{\partial \beta_p} & \frac{\partial T(t_2, \boldsymbol{\beta})}{\partial \beta_p} & \cdots & \frac{\partial T(t_n, \boldsymbol{\beta})}{\partial \beta_p} \end{bmatrix}$$

Variance-covariance matrix

$$\mathbf{cov}\, \hat{\boldsymbol{\beta}} = \sigma^2 \left(\mathbf{X}^{\mathrm{T}} \mathbf{X} \right)^{-1}$$

- Minimum
- Noise assumptions dependent

Variance-covariance matrix

$$\mathbf{cov}\,\hat{\boldsymbol{\beta}} = \sigma^2\left(\mathbf{X}^T\mathbf{X}\right)^{-1}$$

$$\left|\rho_{ij}\right| = \left|\frac{\mathrm{cov}(\hat{\beta}_i,\hat{\beta}_j)}{\sqrt{\sigma^2_{\hat{\beta}_i}\sigma^2_{\hat{\beta}_j}}}\right|$$

$$\mathbf{cov}(\hat{\boldsymbol{\beta}}) \approx \begin{bmatrix} \mathrm{var}(\hat{\beta}_i) & \mathrm{cov}(\hat{\beta}_i,\hat{\beta}_j) & \cdots \\ \mathrm{cov}(\hat{\beta}_i,\hat{\beta}_j) & \mathrm{var}(\hat{\beta}_j) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$\mathbf{cor}(\hat{\boldsymbol{\beta}}) \approx \begin{bmatrix} 1 & \rho_{ij} & \cdots \\ \rho_{ij} & 1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$\mathbf{Vcor}(\hat{\boldsymbol{\beta}}) \approx \begin{bmatrix} \sqrt{\mathrm{var}\left(\hat{\beta}_i\right)}/\hat{\beta}_i & \rho_{ij} & \cdots \\ \rho_{ij} & \sqrt{\mathrm{var}\left(\hat{\beta}_j\right)}/\hat{\beta}_j & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Correlation coefficients

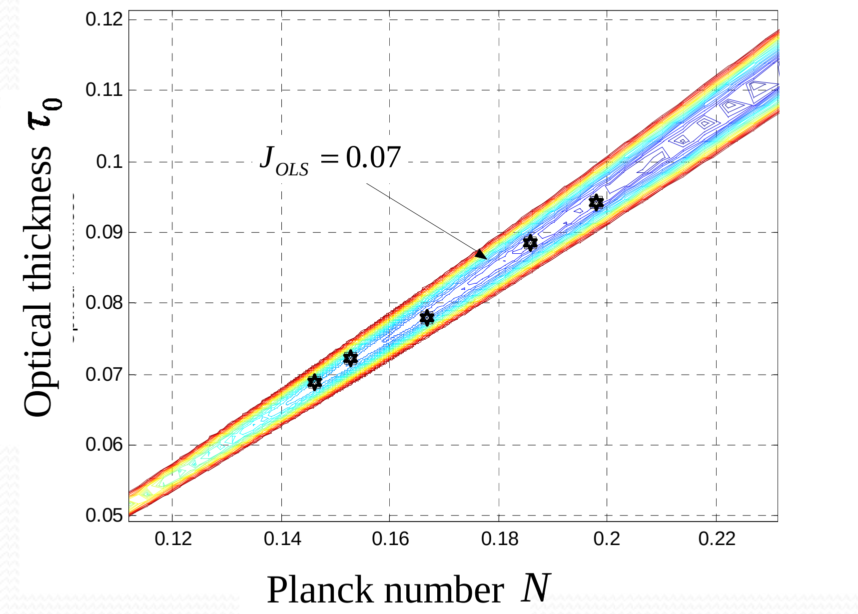
relative error on β_i

TOOL Nr2: Matrix $V_{cor}(\hat{\beta})$ gives a quantitative point of view about the identifiability of the parameters. The diagonal gives a kind of measurement (minimal bound!) of the error made on the estimated parameters (due to the sole stochastic character of the noise, supposed unbiased). The off-diagonal terms (correlation coefficients) are generally of poor interest because of their too global character. Values very close to ± 1 may explain very large variances (errors) on the parameters through a correlation effect.

Ill-Conditioned PEP and Stategies for Tracking True Degrees of Fredom

- Pathological example of ill-conditioning resulting from correlated parameters

The thermal characterization of a semi-transparent material implies at least three basic parameters: the thermal diffusive characteristic time $t_d = e^2/a$, the dimensionless optical thickness τ_0 and the dimensionless Planck number N and so $\boldsymbol{\beta} = [t_d, \tau_0, N]^T$.



Level sets for $J_{OLS}(\boldsymbol{\beta})$

in the (τ_0, N) *parameter space*

Parameter vector components	Local Minima (found using either deterministic or stochastic algorithms)			
	N°1	N°2	N°3	N°4
a (10^7 m²/s)	5.2	4.9	5.85	4.8
N	0.6	0.74	0.16	0.82
τ_0	0.38	0.5	0.07_6	0.56
$R_r = \frac{N_{Pl}}{\tau_0}(\tau_0 + 1)$	2.18	2.22	2.26	2.28

Example of local minima found $\hat{\boldsymbol{\beta}}$

TOOL Nr3: In given conditions of noise and for a given model, it may be interesting to look at the level-set representation of the optimisation criterium in appropriate cut-planes (for given pair of parameters if n>3), and compare it with the minimum achievable criterium given by $J = m\sigma^2$ **.**

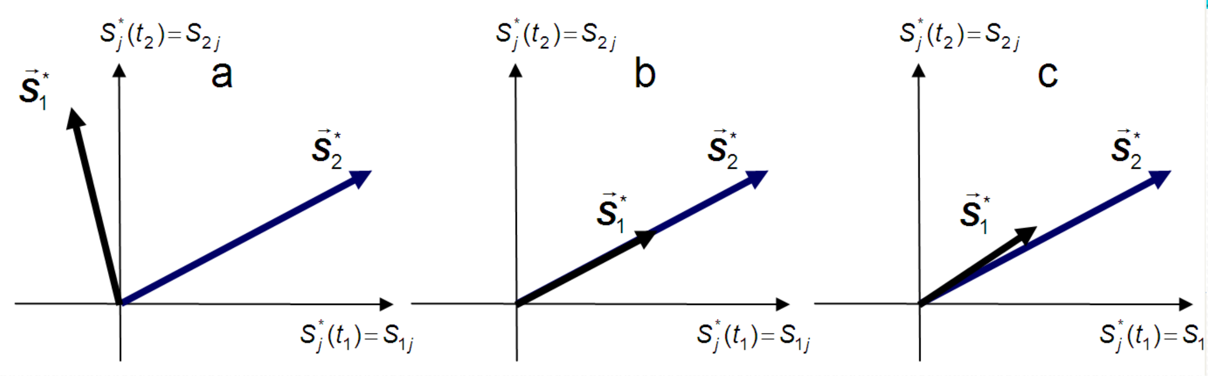
- Rank of the sensitivity Matrix

We focus here on the reduced sensitivity matrix. This (m, n) matrix is composed of n column vectors, the reduced sensitivity coefficients $\boldsymbol{S}_j^*$

$$\boldsymbol{S}^* = \begin{bmatrix} \boldsymbol{S}_1^* & \boldsymbol{S}_2^* & \cdots & \boldsymbol{S}_n^* \end{bmatrix} \qquad \text{with} \quad \boldsymbol{S}_j^* = \beta_j \left. \frac{\partial \boldsymbol{\eta}(\boldsymbol{t};\boldsymbol{\beta}^{nom})}{\partial \beta_j} \right|_{\boldsymbol{t},\, \beta_k \text{ pour } k \neq j}$$

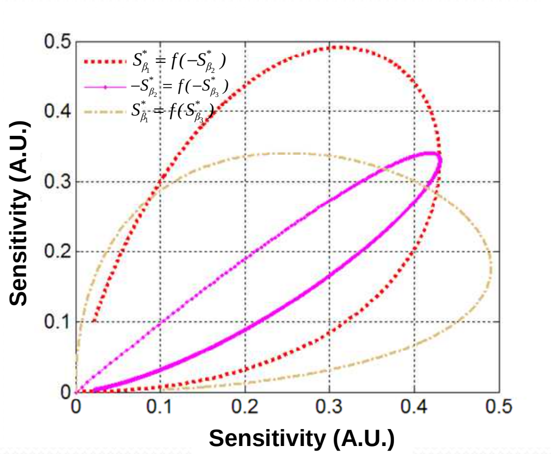
These *n* column vectors $\boldsymbol{S}_j^*$ are in fact just the components of a set of *n* vectors $\vec{\boldsymbol{S}}_j^*$ in a *m*-dimension vector space. One can recall here that this set of vector $\Sigma = \{\vec{\boldsymbol{S}}_1^*, \vec{\boldsymbol{S}}_2^*, \ldots, \vec{\boldsymbol{S}}_n^*\}$ is linearly independent only if:

$$\sum_{j=1}^{n} \alpha_j \boldsymbol{S}_j^* = \boldsymbol{0} \;\Rightarrow\; \alpha_j = 0 \text{ for any } j \text{ such as } 1 \le j \le n$$

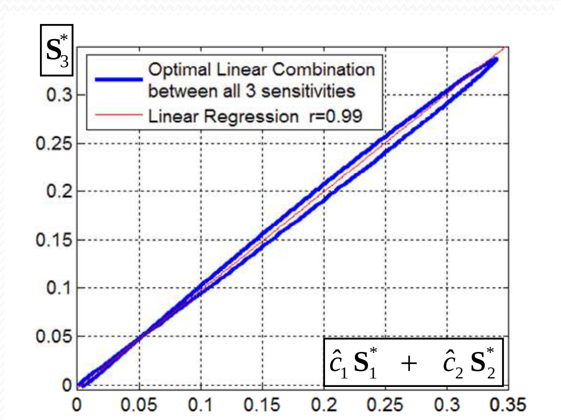


a - independent sensitivities ($r = n = 2$) **b** - dependent sensitivities **c-** nearly dependent sensitivities

Reduced sensitivity vectors



Sensitivities plotted by pairs



Evidence of Linear Combination between all three parameters

TOOL Nr4: The SVD of the normalized sensitivity matrix around nominal values of the parameter vector β can be advantageously calculated to get valuable information.

• Residuals Analysis and Signature of the Presence of a Bias in the Metrological Process

One way to analyse the results of the estimation process is to calculate the residuals (equation 10) at convergence. When equation (8) is checked, it can be easily shown that the expectancy of the residuals curve $\mathbf{r}(\mathbf{t},\hat{\boldsymbol{\beta}})$ is equal to a null function:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}\left[y_i - y_{mo}\left(t_i,\hat{\boldsymbol{\beta}}\right)\right] = \mathbf{E}\left[\mathbf{S}\left(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}}\right)\right] = \mathbf{E}\left[-\mathbf{S}\left(\mathbf{S}^T\mathbf{S}\right)^{-1}\mathbf{S}^T\boldsymbol{\varepsilon}\right] = -\mathbf{S}\left(\mathbf{S}^T\mathbf{S}\right)^{-1}\mathbf{S}^T\mathbf{E}(\boldsymbol{\varepsilon})$$

Since $\mathbf{E}(\boldsymbol{\varepsilon}) = \mathbf{0}$, $\mathbf{E}(\mathbf{r}) = \mathbf{0}$ which means that if the model used for describing the experiment is adapted, the residuals curve is "unsigned" (unbiased theoretical model). On the contrary, "signed" residuals can be considered as the manifestation of some biased estimation.

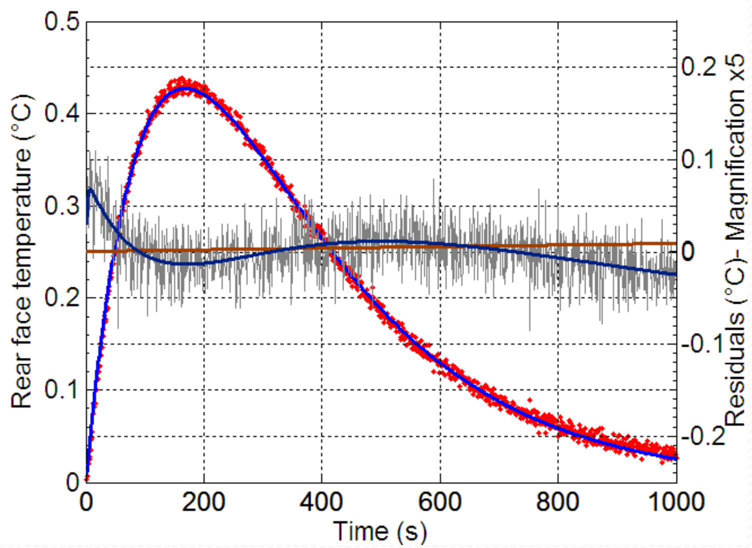
The bias can originates from different sources and mainly:

(i) the a priori decision that some parameters of the model are known and therefore fixed at some given value (maybe measured by another experiment). As authentic parameters of the PEP, they can alter the estimates of the remaining unknown parameters.

(ii) Experimental imperfections which makes the model idealized with respect to the reality of the phenomena.

The existence of a bias means that there exists a systematic and generally unknown inconsistency between the model and the experimental data.

An artificial bias is introduced under the form of a linear drift superimposed to the output simulated observations. It corresponds practically to a linear deviation of the signal from the equilibrium situation before the experiment starts. A noise respecting is also added to the simulation of the measurements so that we have:



Signed character of "post-estimation" residuals in the presence of a bias and using a badly conditioned PEP

$$\mathbf{y} = \eta(\mathbf{t}, \boldsymbol{\beta}) + b(\mathbf{t}) + \boldsymbol{\varepsilon}$$

Time Interval	70 s	150 s	300 s
a (m²/s)	3.76.10-6	3.22.10-6	2.21.10-6
λ (W/m.°C)	0.031	0.064	0.084

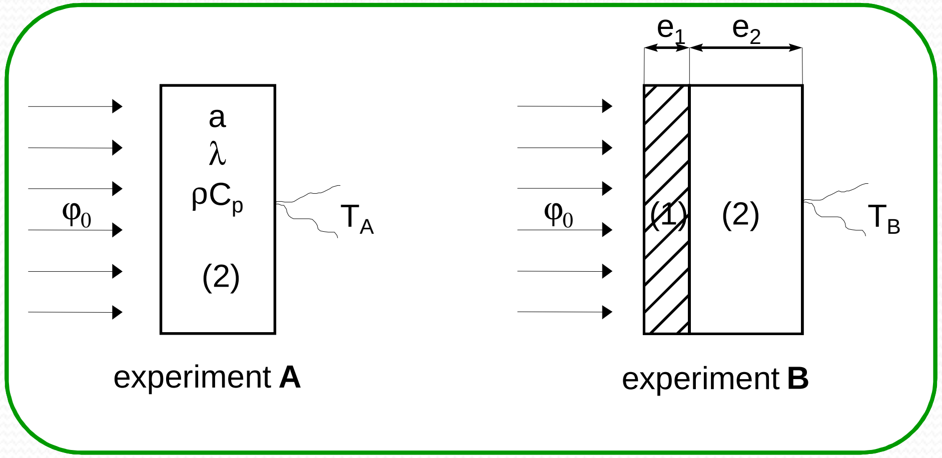
Influence of the existence of some bias on the parameter estimates for a badly conditioned problem

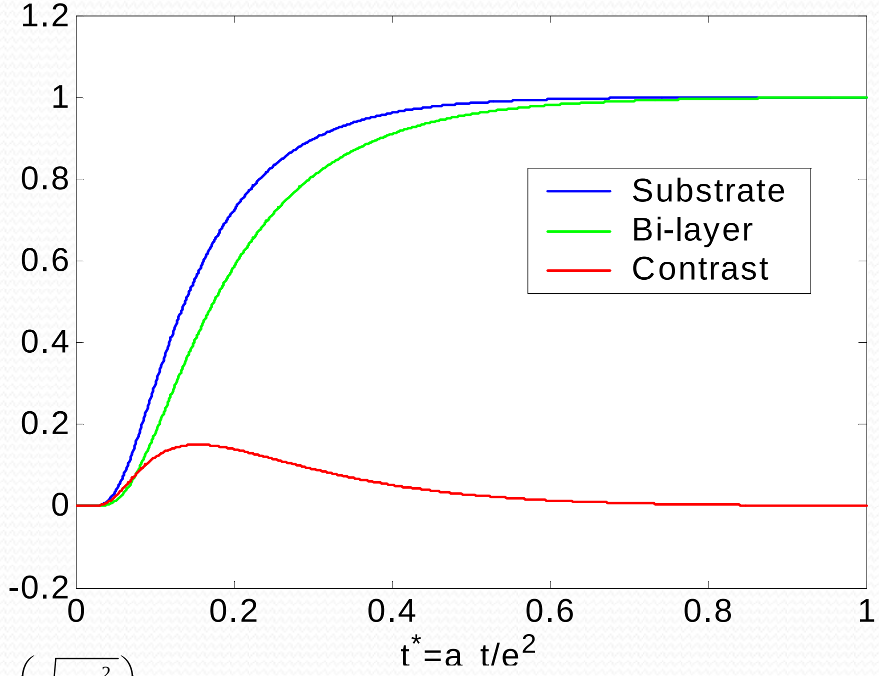
TOOL Nr5: The "post-estimation"residuals have to be analysed carefully to check the instance of a bias of systematic origin. Its magnitude can be compared to the standard deviation of the white noise of the sensor to check whether this bias may introduce too large confidence intervals of the estimates (with respect to the pure stochastic estimation of the variances of parameter estimates in the absence of any bias). Relative invariance of the estimates with respect to the identification intervals may suggest that the bias is acceptable. In the opposite case, strategies must begin either to change the nature of the estimation problems (reduce initial goals) or to use residuals to give a fait quantitative evaluation of confidence bounds of the estimates.

Reducing the PEP to Make It Well-Conditionned & Dimensional Analysis

Case of the Contrast Method

Measurement of the thermophysical properties of deposits by the bilayer or "thermal contrast" technique





$$\begin{bmatrix}\theta_{i_{in}} \\ \phi_{i_{in}}\end{bmatrix} = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix}\theta_{i_{out}} \\ \phi_{i_{out}}\end{bmatrix}$$

$$A_i = D_i = \cosh\left(\sqrt{\frac{pe_i^2}{a_i}}\right),$$

$$B_i = \frac{1}{\lambda\lambda_i\sqrt{\frac{p}{a_i}}}\sinh\left(\sqrt{\frac{pe_i^2}{a_i}}\right) \text{ et } C_i = \lambda\lambda_i\sqrt{\frac{p}{a_i}}\sinh\left(\sqrt{\frac{pe_i^2}{a_i}}\right)$$

Laplace Transform

$$\theta(z,p) = \mathcal{L}(T(z,t)) = \int_0^\infty T(z,t)\exp(-pt)dt$$

Reduced Laplace Transform

$$\tilde{\theta}(z,p^*) = \tilde{\mathcal{L}}(T(z,t^*)) = \int_0^\infty T(z,t^*)\exp(-p^*t^*)\,dt^*$$

$$t^* = \frac{a_2 t}{e_2^2} \qquad s = \sqrt{p^*} \qquad p^* = p\frac{e_2^2}{a_2}$$

➢ Flash Experiment on the substrate:

$$\begin{bmatrix} \theta_{2in} \\ \phi_{2int} = \varphi_{02} \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} \theta_{2out} \\ \phi_{2out} = 0 \end{bmatrix}$$

$$\theta_{2out} = \frac{\varphi_{0_2}}{C_2} = \frac{\varphi_{02}}{\lambda_2 \sqrt{\frac{p}{a_2}} \sinh\left(\sqrt{\frac{pe_2^{\,2}}{a_2}}\right)}$$

$$\tilde{\theta}^*_{2_{out}} = \tilde{\mathfrak{L}}\left(\frac{T_{2_s}}{T_{2_\infty}}\right) = \frac{1}{s\ \sinh(s)}$$

➢ Flash Experiment on the bi-layer material:

$$\begin{bmatrix} \theta_{1/2in} \\ \phi_{1/2in} = \varphi_{0_{1/2}} \end{bmatrix} = \begin{bmatrix} A_{eq} & B_{eq} \\ C_{eq} & D_{eq} \end{bmatrix} \begin{bmatrix} \theta_{1/2out} \\ \phi_{1/2out} = 0 \end{bmatrix}$$

With:
$$\begin{bmatrix} A_{eq} & B_{eq} \\ C_{eq} & D_{eq} \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1A_2 + B_1C_2 & A_1B_2 + A_2B_1 \\ A_1C_2 + A_2C_1 & A_1A_2 + B_2C_1 \end{bmatrix}$$

$$A_{eq} \neq D_{eq}$$

$$\theta_{1/2s} = \frac{\varphi_{0_{1/2}}}{C_{eq}} = \frac{\varphi_{01/2}}{A_1C_2 + A_2C_1}$$

and

$$\theta_{1/2out} = \frac{\varphi_{01/2}}{\lambda_1 \sqrt{\frac{p}{a_1}} \sinh\left(\sqrt{\frac{pe_1^{\,2}}{a_1}}\right) \cosh\left(\sqrt{\frac{pe_2^{\,2}}{a_2}}\right) + \lambda_2 \sqrt{\frac{p}{a_2}} \sinh\left(\sqrt{\frac{pe_2^{\,2}}{a_2}}\right) \cosh\left(\sqrt{\frac{pe_1^{\,2}}{a_1}}\right)}$$

$$T_{1/2_\infty} = \frac{\varphi_{0_{1/2}}}{\rho_1 c_1 e_1 + \rho_2 c_2 e_2}$$

$$\theta^*_{1/2_{out}} = \frac{e_2^2}{a_2} \frac{1 + \frac{\rho_1 c_1 e_1}{\rho_2 c_2 e_2}}{s\left[\sqrt{\frac{\lambda_1 \rho_1 c_1}{\lambda_2 \rho_2 c_2}} \sinh\left(\frac{e_1}{e_2}\sqrt{\frac{a_2}{a_1}} s\right) \cosh(s) + \sinh(s) \cosh\left(\frac{e_1}{e_2}\sqrt{\frac{a_2}{a_1}} s\right)\right]}$$

Let introduce now the parameters:

$$K_1 = \frac{e_1}{e_2}\sqrt{\frac{a_2}{a_1}}$$

ratio of the root of characteristic times

(depends on the thicknesses of the materials)

$$K_2 = \sqrt{\frac{\lambda_1 \rho_1 c_1}{\lambda_2 \rho_2 c_2}}$$

ratio of the thermal effusivities

(intrinsic to the nature of the two layers)

$$\tilde{\theta}^*_{1/2_{out}} = \frac{1}{s}\left[\frac{1+K_1K_2}{K_2\sinh(K_1 s)\cosh(s)+\sinh(s)\cosh(K_1 s)}\right]$$

- **Contrast curve:** $\Delta\tilde{\theta}^*_{out} = \tilde{\theta}^*_{1/2_{out}} - \tilde{\theta}^*_{2_{out}} = \tilde{\mathfrak{L}}\left(T^*_{1/2_{out}} - T^*_{2_{out}}\right) = \tilde{\mathfrak{L}}\left(\Delta T^*\right)$

$$\Delta\tilde{\theta}^*_{\text{out}} = \frac{1}{s}\left[\frac{1+K_1K_2}{K_2\sinh(K_1 s)\cosh(s)+\sinh(s)\cosh(K_1 s)} - \frac{1}{\sinh(s)}\right]$$

$$K_3 = K_1\,K_2 = \frac{\rho_1 c_1 e_1}{\rho_2 c_2 e_2}$$

thermal capacities ratio

$$K_4 = \frac{K_1}{K_2} = \frac{e_1}{e_2}\frac{\lambda_2}{\lambda_1}$$

thermal resistances ratio

In all cases, the corresponding substrate properties must to be known

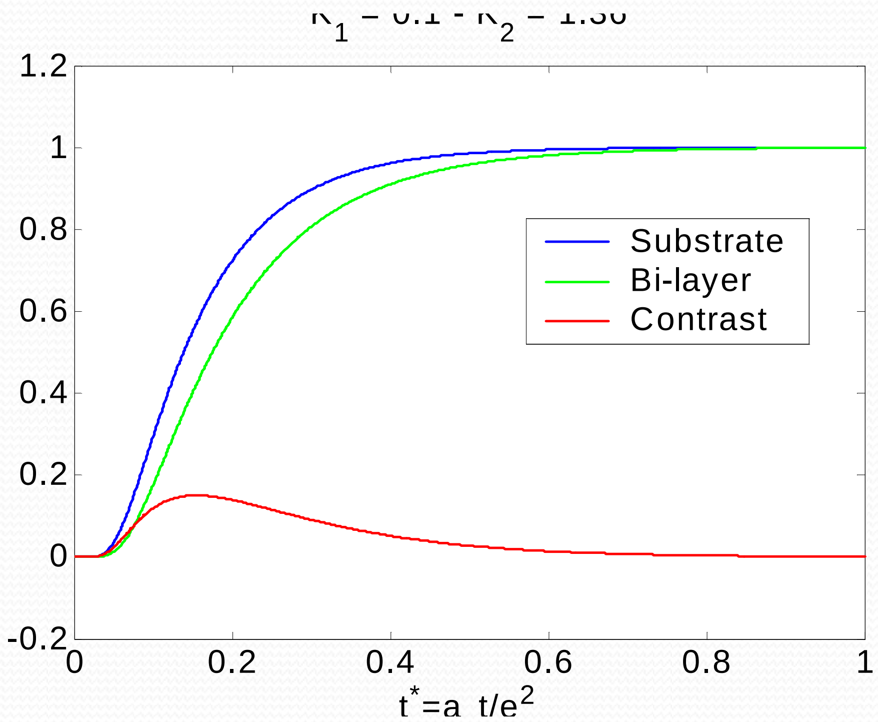
Case 1 :

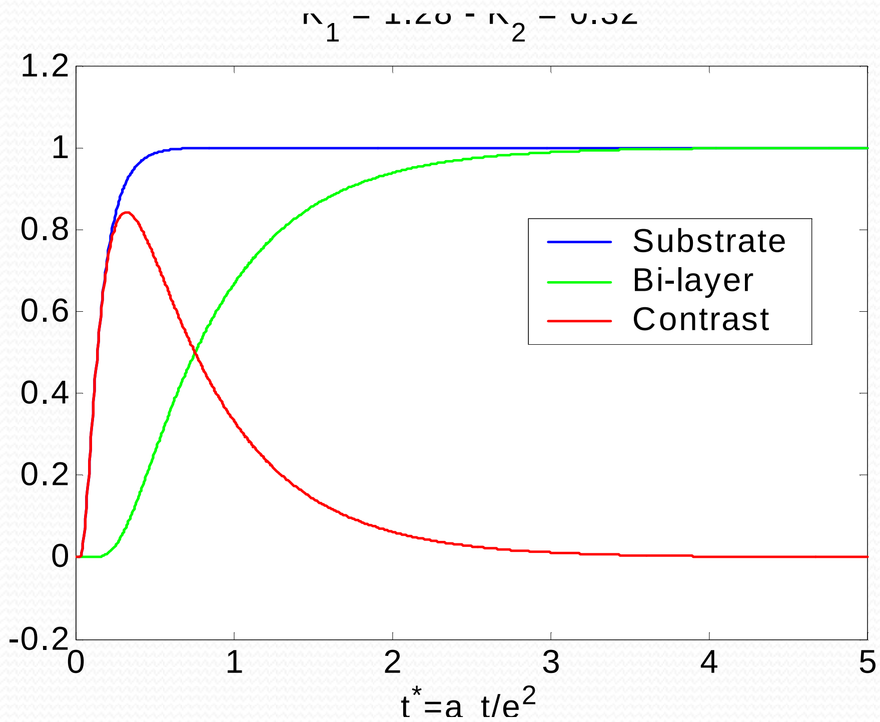
Conductive coating / Insulating substrate

	Thickness (µm)	a (m^2/s)	λ (W/m.°K)	ρC_p ($J/m^3.°K$)
Case 1 :	**Aluminium coating on a Cobalt/Nickel substrate**			
Film (1)	220	$9{,}46.10^{-5}$	230	$2{,}43.10^{6}$
Substrate (2)	1 100	$2{,}36.10^{-5}$	84,5	$3{,}57.10^{6}$
$h = 10\ W/m^2.°C$	$Bi = \frac{he_2}{\lambda_2} = 1{,}3.10^{-4} << 1$			
Case 2 :	**Insulating film on a Alumina substrate**			
Film (1)	247	$6{,}84.10^{-7}$	2,23	$3{,}26.10^{6}$
Substrate (2)	640	$7{,}47.10^{-6}$	23	$3{,}08.10^{6}$
$h = 10\ W/m^2.°C$	$Bi = \frac{he_2}{\lambda_2} = 2{,}8.10^{-4} << 1$			

Case 2 :

Insulating film / Conductive substrate

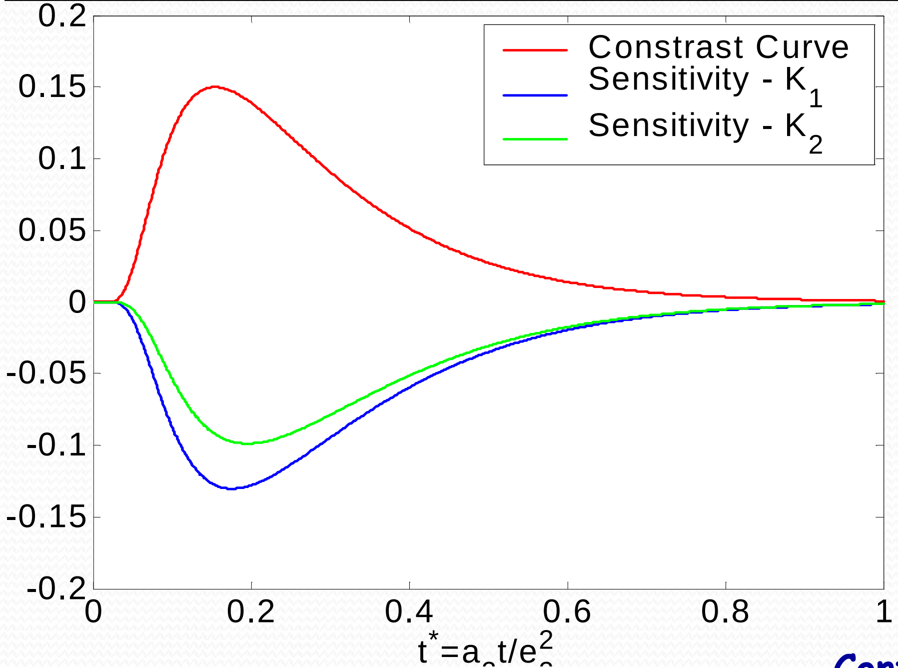




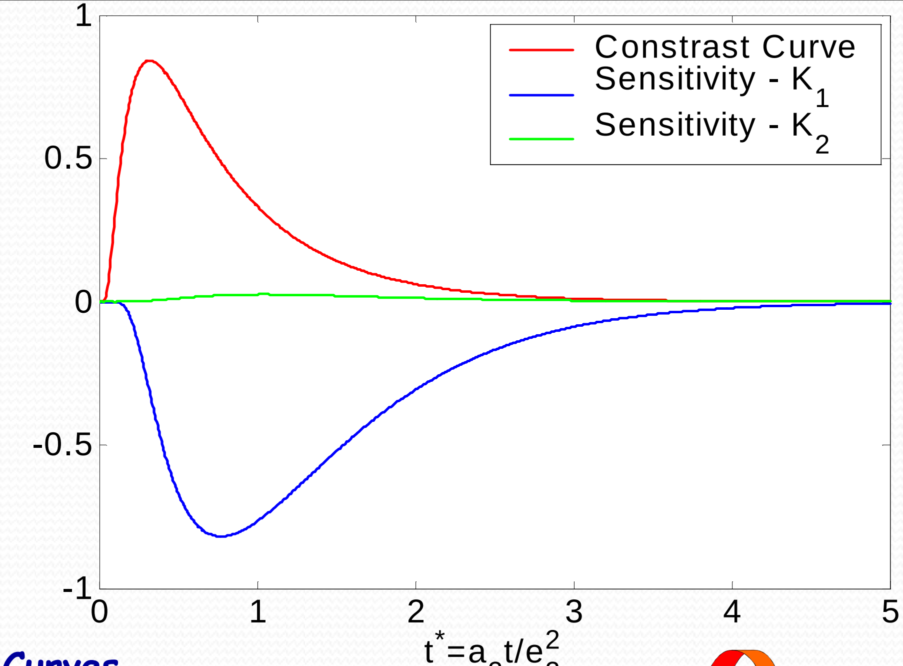
Sensitivity Study Through 2 Examples

$$Cov(K) = \sigma_c^{\,2} \left(X^t X \right)^{-1}$$

Case 1 : Conductive coating / Insulating substrate



Case 2 : Insulating coating / Conductive substrate



Contrast Curves

Case 1	$\sigma_N = 1$ / $Npt = 1000$	Case 2
Variance-Covariance **28.0302** -35.9846 -35.9846 **46.6417**	$\sigma_N = 1$	**Variance-Covariance** **0.1067** 3.1409 3.1409 **99.1677**
Correlation 1.0000 -0.9952 -0.9952 1.0000	$Npt = 1000$	**Correlation** 1.0000 0.9655 0.9655 1.0000

K_1 is closed to unity

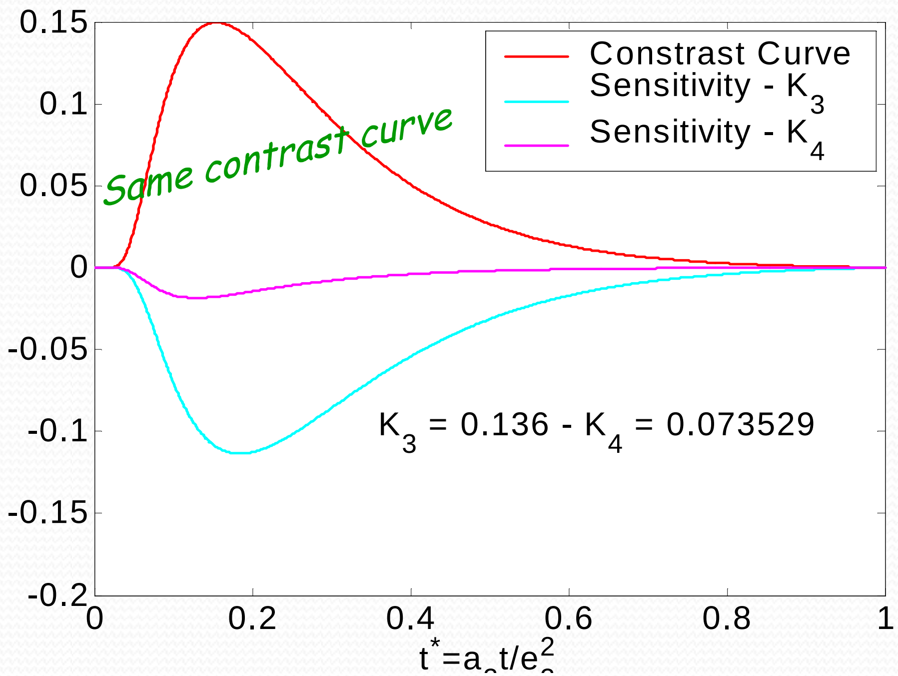
$$\sinh(K_1 s)\cosh(s) \simeq K_1 \sinh(s)\cosh(K_1 s)$$

$$\Delta\tilde{\theta}^*_{out} = \frac{1}{s}\left[\frac{1}{\sinh(s)\cosh(K_1 s)} - \frac{1}{\sinh(s)} \right]$$

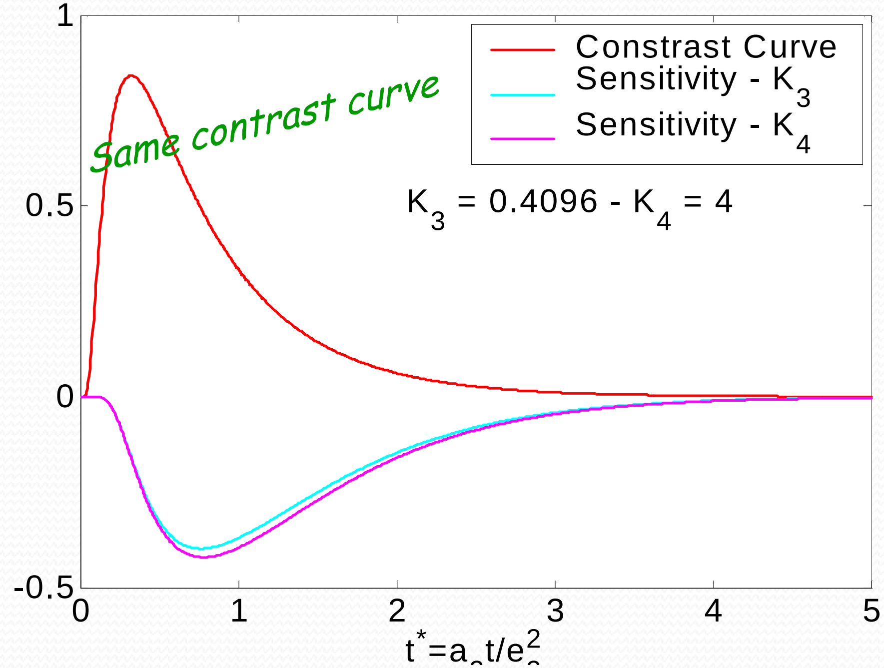
Covariance & Correlation Matrices

Introduction of a new set of parameters : (K_3, K_4)

Case 1 : *Conductive coating / Insulating substrate*



Case 2 : *Insulating coating / Conductive substrate*



Sensitivity Curves

Variance-Covariance	
2.6921	-18.5189
-18.5189	**145.8475**
Correlation	
1.0000	-0.9346
-0.9346	1.0000
Case 1	

K_1 is closed to zero

Variance-Covariance	
103.5845	-97.1801
-97.1801	**91.1985**
Correlation	
1.0000	-0.9999
-0.9999	1.0000
Case 2	

$$\begin{cases} \sinh(K_1 s) \simeq K_1 s \\ \cosh(K_1 s) \simeq 1 \end{cases}$$

$$\Delta\tilde{\theta}_s^* = \frac{1}{s}\left[\frac{1 + K_1 K_2}{K_2 \sinh(K_1 s)\cosh(s) + \sinh(s)\cosh(K_1 s)} - \frac{1}{\sinh(s)}\right]$$

$$\Delta\tilde{\theta}_s^* = \frac{1}{s}\left[\frac{1 + K_3}{K_3 s \cosh(s) + \sinh(s)} - \frac{1}{\sinh(s)}\right]$$

Example

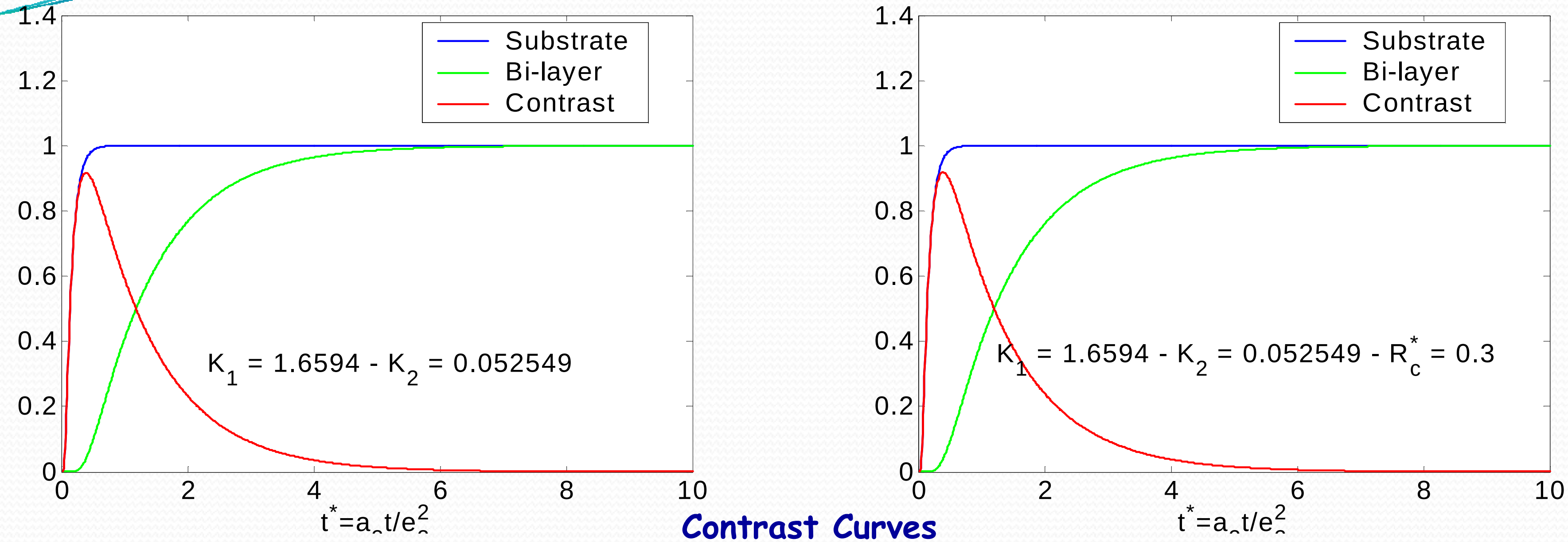
Bi-layer material : *P.V.C deposit / Steel substrate*

	Thickness (mm)	a (m^2/s)	λ (W/m.°K)	ρC_p ($J/m^3.°K$)
Case:	**P.V.C deposit on a Steel substrate**			
Film (1)	1	$1,21.10^{-7}$	0.19	$1,57.10^{6}$
Substrate (2)	5	$8,33.10^{-6}$	30	$3,60.10^{6}$
Nominal values	$K_1 = 1.66$	$K_2 = 0.052$	$K_3 = 0.086$	$K_4 = 31.92$
$R_c = 5.10^{-5}\ K/W.m^2$	$R_c^* = \dfrac{R_c}{(e_2/\lambda_2)} = 0,3$			

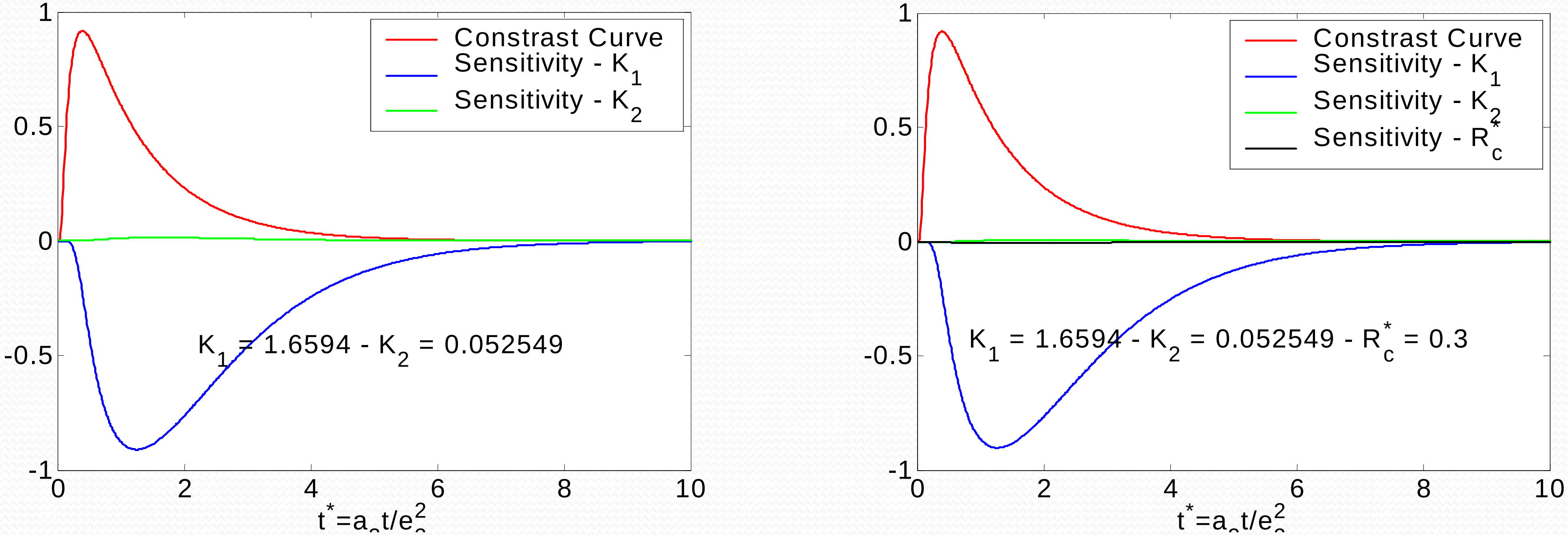
Case 2: Insulating coating / Conductive substrate

Example

Bi-layer material : P.V.C deposit / Steel substrate



Contrast Curves

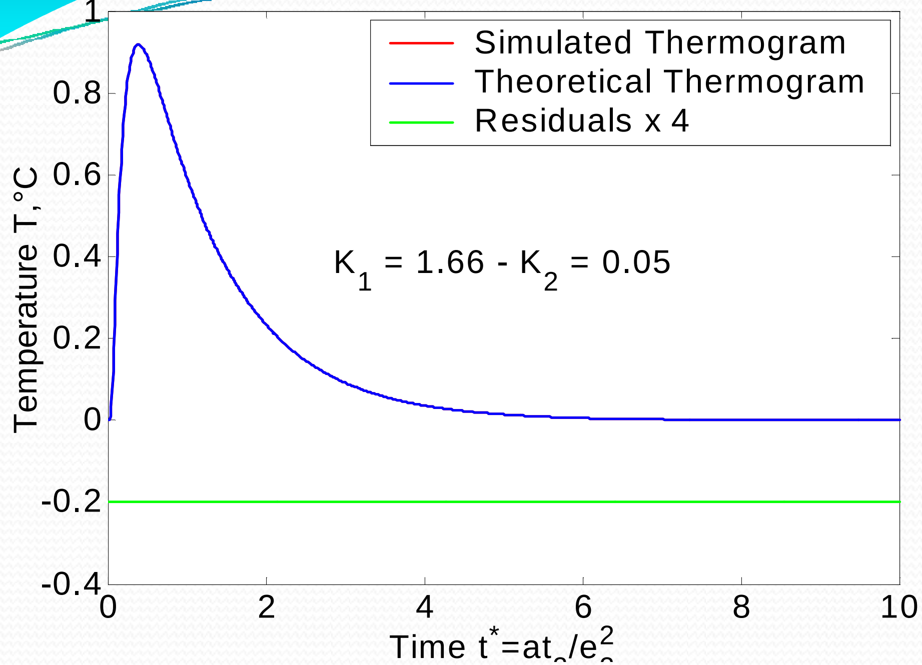


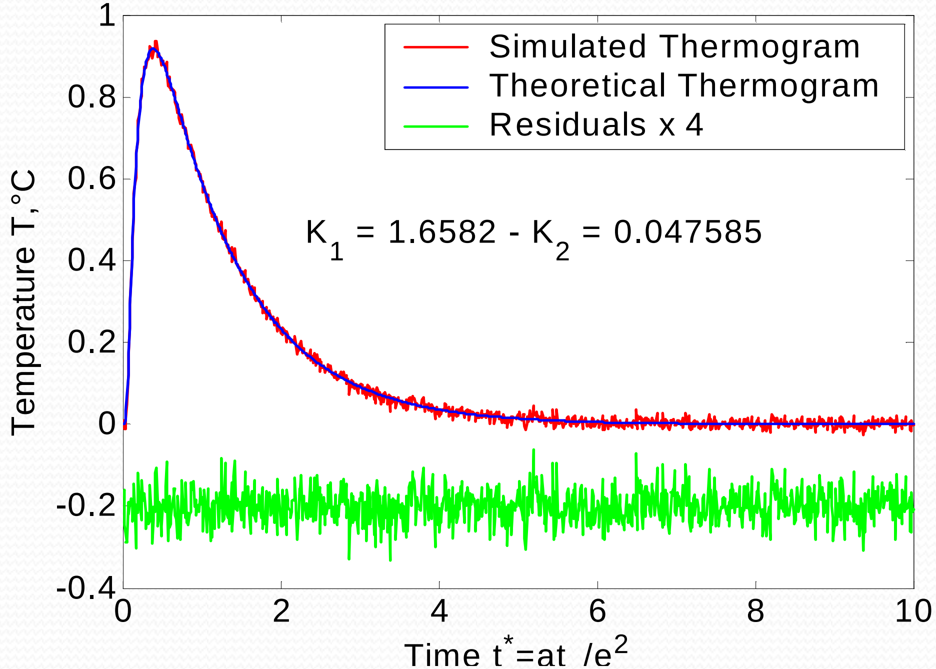
Sensitivity Curves

Example

Nominal Values : $K_1 = 1.66 - K_2 = 0.05$

(Perfect Contact)





Estimation without noise

Variance-Covariance	
$\sigma_N^2 x$	
0.1670	10.9443
10.9443	**745.4980**
Correlation	
1.0000	0.9809
0.9809	1.0000

$K_1 = 1.6582 \pm 0.0067$

$\frac{\sigma_{K_1}}{K_1} = \sigma_N \sqrt{\mathrm{cov}^*(K_1)} = 0.41\%$

$K_2 = 0.047585 \pm 0.013$

$\frac{\sigma_{K_2}}{K_2} = \sigma_N \sqrt{\mathrm{cov}^*(K_2)} = 27\%$

Estimation with noise

As predicted by theory, without noise we exactly find the nominal values used for the simulation

As predicted by theory, the more sensitive parameter in this case is the parameter K_1

Optimization of the experiment

Can the parameter estimation be improved by a change of parameters

Note on the change of parameters

Let introduce now a new couple of parameters: (K_a, K_b) → $\Delta\theta = f(t, K_1, K_2) = f(t, K_a, K_b)$

The new parameters introduced are function of the old ones: $K_a = F_a(K_1, K_2)$, $K_b = F_b(K_1, K_2)$

$$J = \begin{bmatrix} \frac{\partial F_a}{\partial K_1} & \frac{\partial F_a}{\partial K_2} \\ \frac{\partial F_b}{\partial K_1} & \frac{\partial F_b}{\partial K_2} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

So it is for the Sensivity and Covariance matrices:

Sensitivity $X_{ab} = X_{12}.J^{-1}$ Covariance $\begin{bmatrix} Var(K_a) & Cov(K_a, K_b) \\ Cov(K_a, K_b) & Var(K_b) \end{bmatrix} = J.\begin{bmatrix} Var(K_1) & Cov(K_1, K_2) \\ Cov(K_1, K_2) & Var(K_2) \end{bmatrix}.J^t$

$$\mathrm{Var}(K_a) = a_1^2\,\mathrm{Var}(K_1) + a_2^2\,\mathrm{Var}(K_2) + 2a_1a_2\,\mathrm{Cov}(K_1, K_2)$$
$$\mathrm{Var}(K_b) = b_1^2\,\mathrm{Var}(K_1) + b_2^2\,\mathrm{Var}(K_2) + 2b_1b_2\,\mathrm{Cov}(K_1, K_2)$$
$$\mathrm{Cov}(K_a, K_b) = a_1b_1\,\mathrm{Var}(K_1) + a_2b_2\,\mathrm{Var}(K_2) + (a_2b_1 + a_1b_2)\mathrm{Cov}(K_1, K_2)$$

$K_a = F_a(K_1) = K_1$
$K_b = F_b(K_1, K_2)$

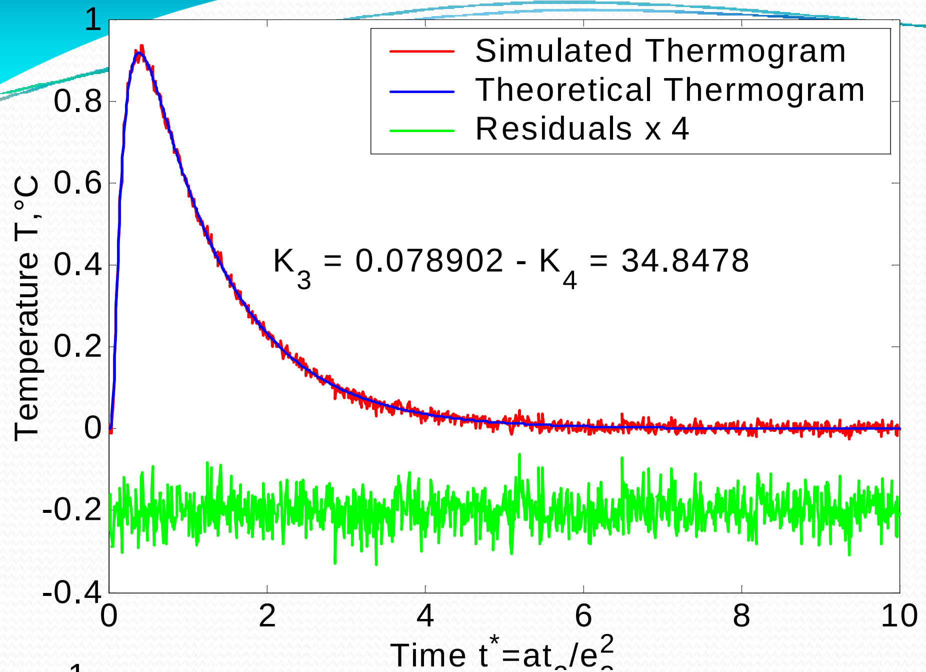
→

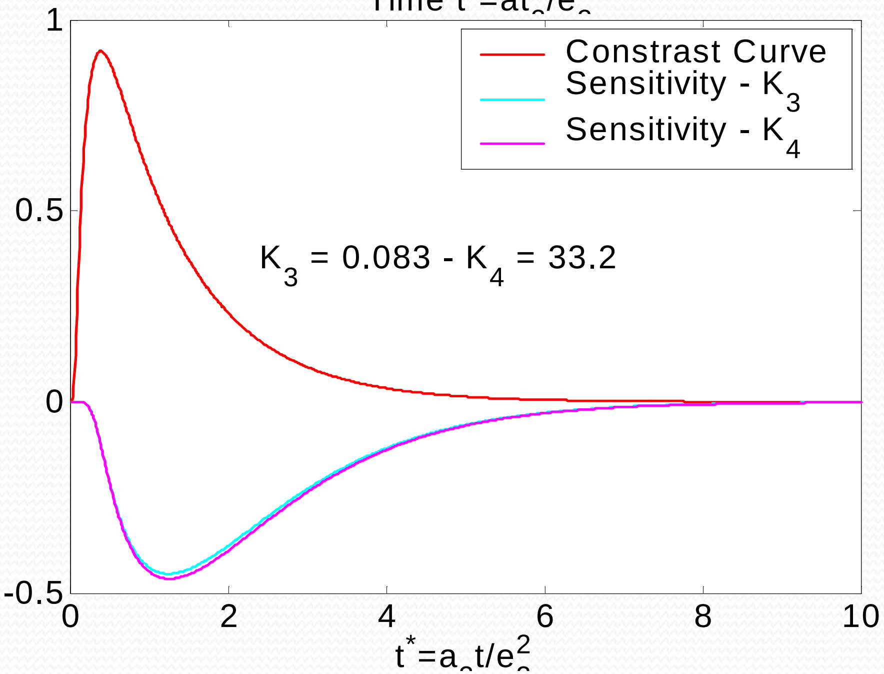
$$\mathrm{Var}(K_a) = \mathrm{Var}(K_1)$$
$$\mathrm{Var}(K_b) = b_1^2\,\mathrm{Var}(K_1) + b_2^2\,\mathrm{Var}(K_2) + 2b_1b_2\,\mathrm{Cov}(K_1, K_2)$$
$$\mathrm{Cov}(K_a, K_b) = b_1\,\mathrm{Var}(K_1) + b_2\,\mathrm{Cov}(K_1, K_2)$$

The **standard-deviation** of a **given parameter** does not depend on the **choice** of the **second parameter**

Let consider now an estimation with: (K_3, K_4)

Nominal Values : $K_3 = 0.086$ – $K_4 = 31.92$





Variance-Covariance 767.6456 -745.4210 -745.4210 723.8643	$K_3 = 0.078902 \pm 0.022$ $\frac{\sigma_{K_3}}{K_3} = \sigma_N \sqrt{\text{cov}^*(K_3)} = 27\%$
Correlation 1.0000 -1.0000 -1.0000 1.0000	$K_4 = 34.8478 \pm 9.3757$ $\frac{\sigma_{K_4}}{K_4} = \sigma_N \sqrt{\text{cov}^*(K_4)} = 27\%$

As expected,
the two parameters are not well estimated

$$K_1 = \sqrt{K_3.K_4} = \sqrt{0,0789.34,848} = 1,6581$$

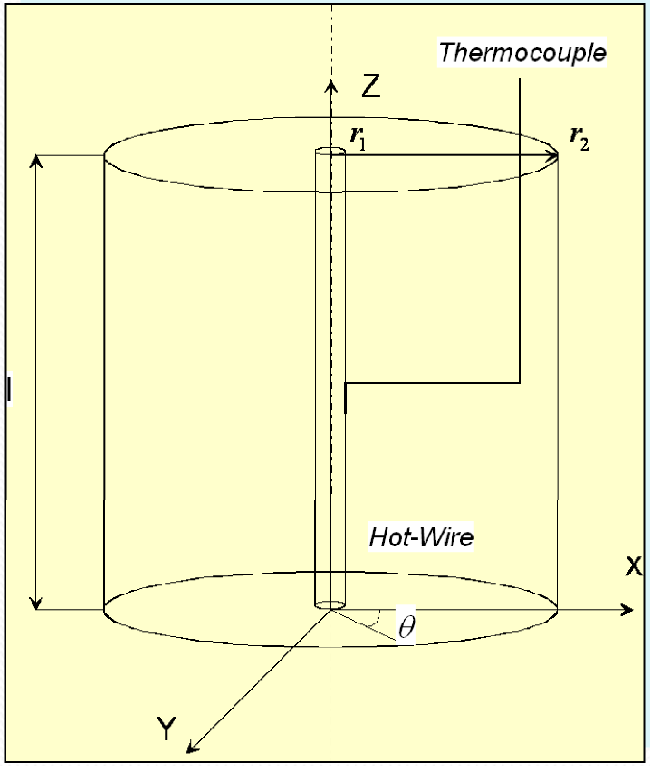
$$K_2 = \sqrt{K_3 / K_4} = \sqrt{0,0789 / 34,848} = 0,0476$$

We exactly find the same values for (K_1, K_2) as those obtained with the estimation in (K_1, K_2)

Over-Parameterized Models

Case of the Hot-Wire Experiment

Principle of the *Hot-Wire* Technique



$0<r<r_1$: Hot-Wire

$r_1<r<r_2$: Medium

- Transient Heat Transfer Equation :

$$div\left(\lambda \cdot \overrightarrow{grad}(T)\right) + P = \rho C_p \frac{\partial T}{\partial t}$$

Cylindrical Coordinate System :

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{P}{\lambda} = \frac{1}{a}\frac{\partial T}{\partial t}$$

- Boundaries Conditions :

$$-\lambda S_1 \left.\frac{\partial T}{\partial r}\right|_{r=r_1} = \Phi_1$$

$$-\lambda S_2 \left.\frac{\partial T}{\partial r}\right|_{r=r_2} = \Phi_2$$

- Initial Condition :

$$T = T_{ext}$$

Assumptions :

- Infinite Expansion
- Azimuthal Symmetry
- Isotropic Medium

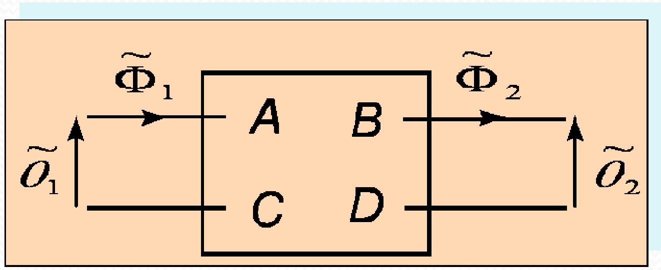
Theoretical Model : *"quadrupole approach"*

- *Laplace* Transform :

$$\tilde{\theta}(r,p)=\int_0^{\infty}\theta(r,t)\exp(-pt)\,dt$$

(numerical inversion)

- *Quadrupole* Formulation :



$$\begin{pmatrix}\tilde{\theta}_1(r_1,p)\\ \tilde{\Phi}_1(r_1,p)\end{pmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{pmatrix}\tilde{\theta}_2(r_2,p)\\ \tilde{\Phi}_2(r_2,p)\end{pmatrix}$$

Transfer Matrix : M

$\rightarrow$ *Waterfall Setting*

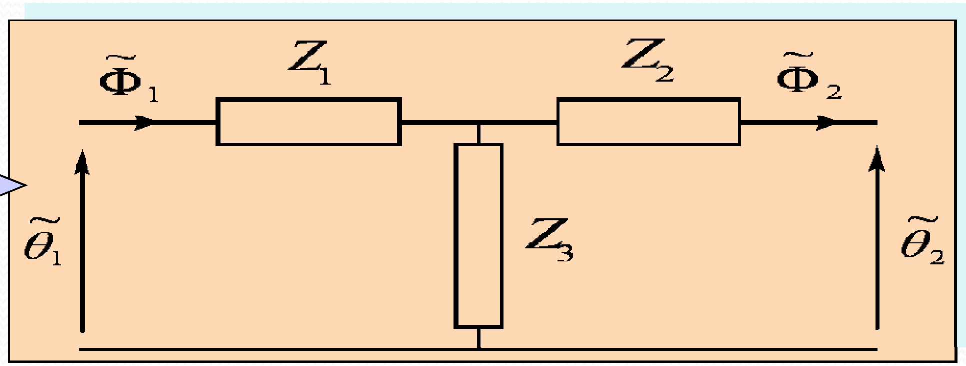
- Representation in a *"T"* - *Quadrupole* Form :

$$A=kr_2\left(K_1(kr_2)\cdot I_0(kr_1)+K_0(kr_1)\cdot I_1(kr_2)\right)$$

$$B=\frac{1}{2\pi\lambda l}\left(K_0(kr_1)\cdot I_0(kr_2)-K_0(kr_2)\cdot I_0(kr_1)\right)$$

$$C=-2\pi\lambda lk^2r_1r_2\left(K_1(kr_2)\cdot I_1(kr_1)-K_1(kr_1)\cdot I_1(kr_2)\right)$$

$$D=kr_1\left(K_0(kr_2)\cdot I_1(kr_1)+K_1(kr_1)\cdot I_0(kr_2)\right)$$



$$Z_1=\frac{A-1}{C},\ Z_2=\frac{D-1}{C}\quad et\quad Z_3=\frac{1}{C}$$

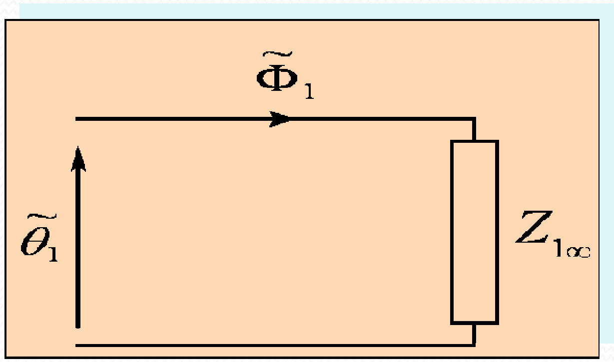
Particular Case : “*semi-infinite medium*”

- *Semi-infinite* Medium : *($r_2 >> 1$)*

$$Z_{1\infty} = \frac{1}{2\pi\lambda l}\frac{K_0(kr_1)}{kr_1 \cdot K_1(kr_1)}$$

$$Z_{2\infty} = \frac{1}{2\pi\lambda l}\frac{1}{kr_2} \to 0$$

$$Z_{3\infty} = \frac{1}{C} \to 0$$



Quadrupole Formulation

- *Asymptotic* Model : *$p \to 0$ et $r_1 << 1$*

Bessel’s Functions *Approximation :*

$$\begin{cases} \lim\limits_{kr_1 \to 0} K_0(kr_1) \simeq -\ln(kr_1) \\ \lim\limits_{kr_1 \to 0} K_1(kr_1) \simeq (2/kr_1)\cdot\Gamma(1)/2 = \dfrac{1}{kr_1} \end{cases}$$

$$\tilde{\theta}_1 = Z_{1\infty}\tilde{\Phi}_1 = -\frac{\ln(kr_1)}{2\pi\lambda l}\tilde{\Phi}_1$$

Response to a Step Stimulation : $\tilde{\Phi}_1 = \dfrac{\Phi_1}{p}$

$$\theta_1(r_1,t) = \frac{\Phi_1}{4\pi l\lambda}\cdot\ln(t) + C^{ste}$$

$$\Delta\theta_1 = \frac{\Phi_1}{4\pi l\lambda}\cdot\ln(t_2/t_1)$$

Non-ideal Aspects

- **Hot-Wire *effects* :**

New definition of inner parameters :

$$\theta_m = \frac{1}{V_1}\int_0^{r_1} \theta \cdot 2\pi r l \, dr = \frac{2}{r_1^{\,2}}\int_0^{r_1} \theta \cdot r \, d$$

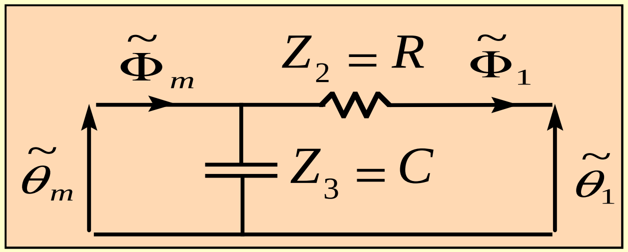
$$\Phi_m = V_1 \cdot G_0(p)$$

Asymptotic expansion : p→ 0 et r_1<<1

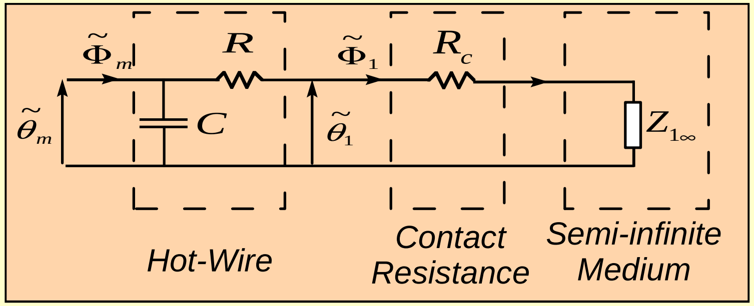
$$Z_1 = 0$$

$$Z_2 = \frac{1}{8\pi\lambda l} \quad \rightarrow \quad \text{Resistance}$$

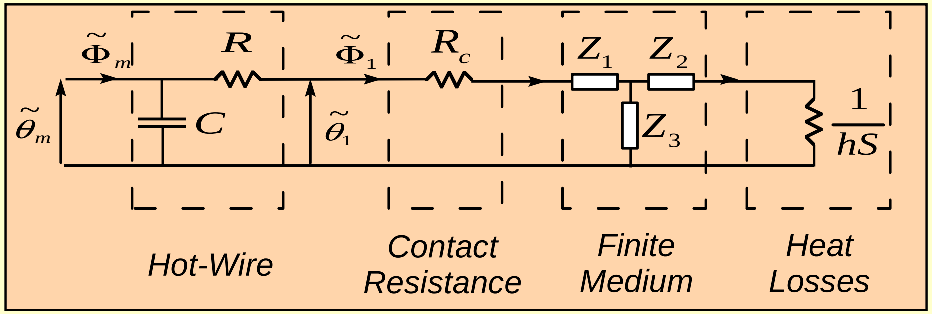
$$Z_3 = \frac{1}{\pi r_1^{\,2} l \cdot \rho C_p \cdot p} \quad \rightarrow \quad \text{Capacity}$$



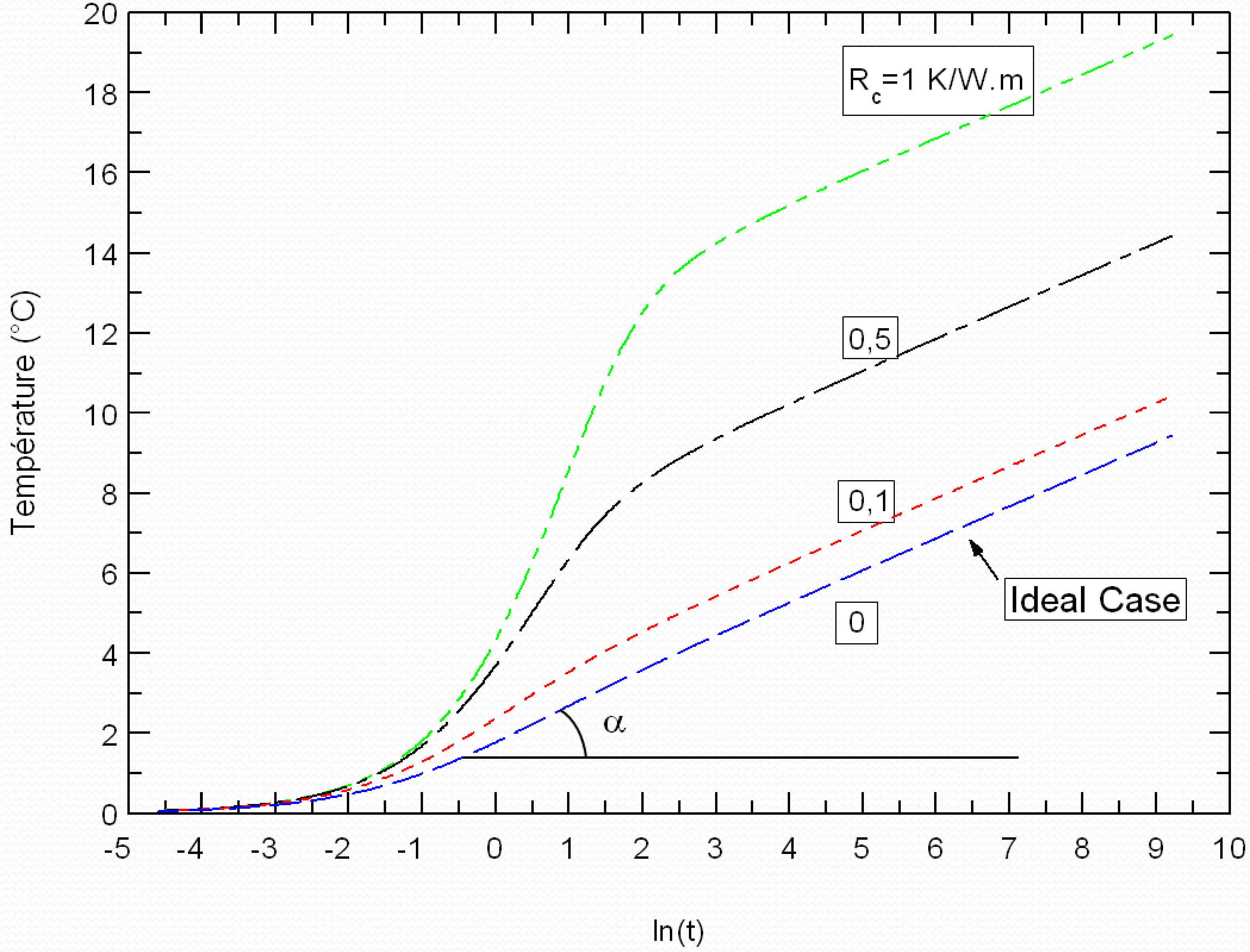
- ***Contact* Resistance :**



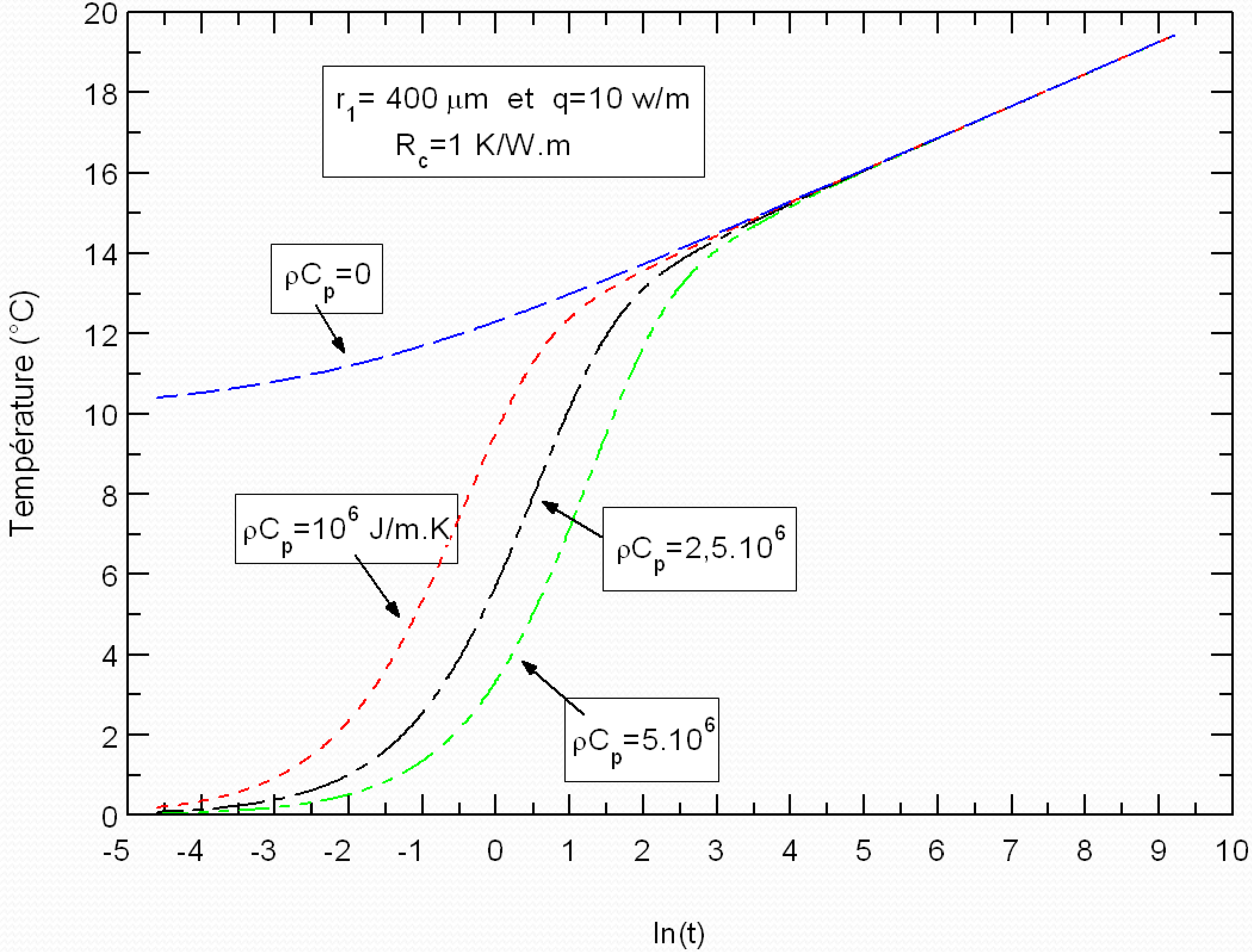
- ***Finite* Medium (Heat Losses Effects) :**



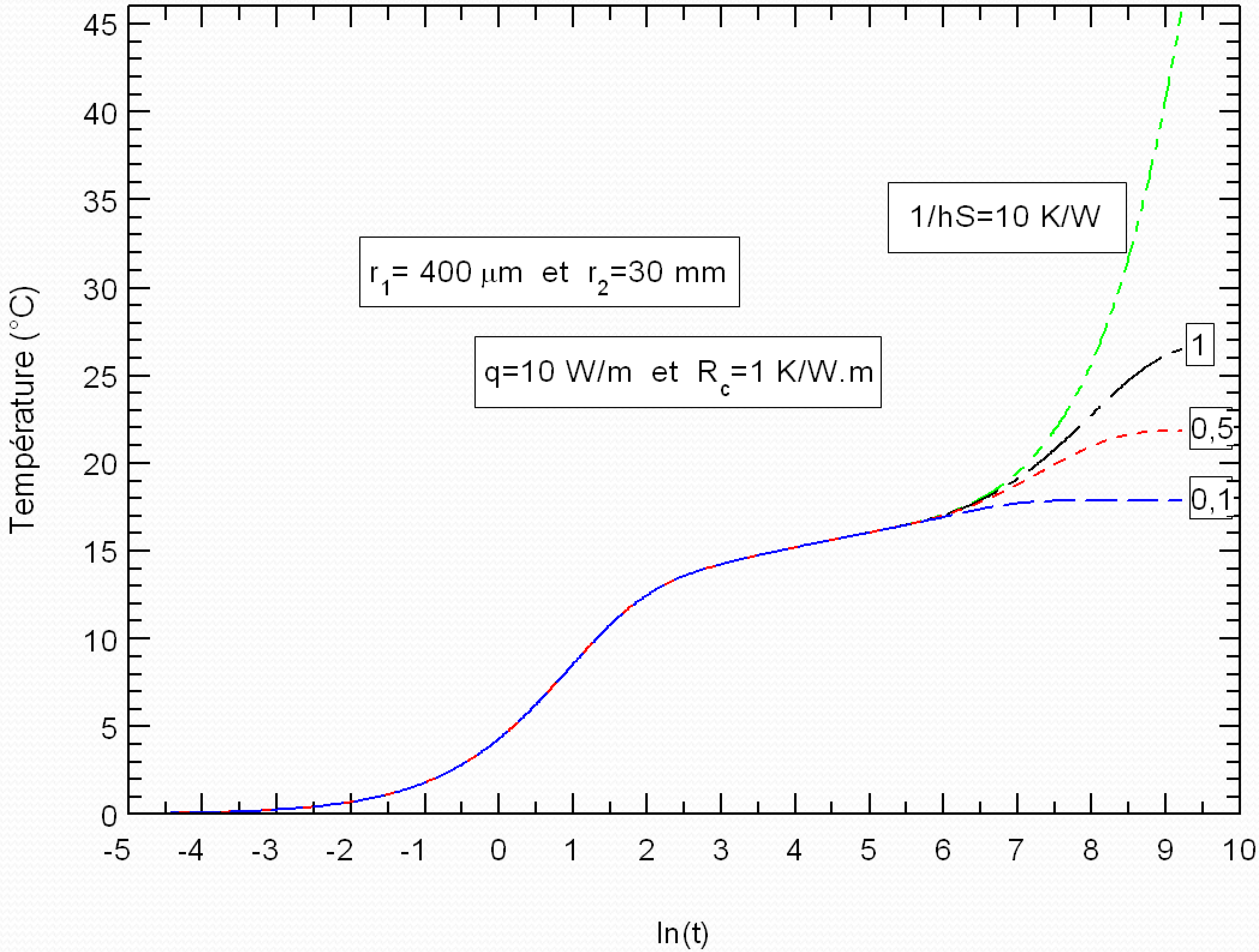
Effect of the *Contact Resistance*



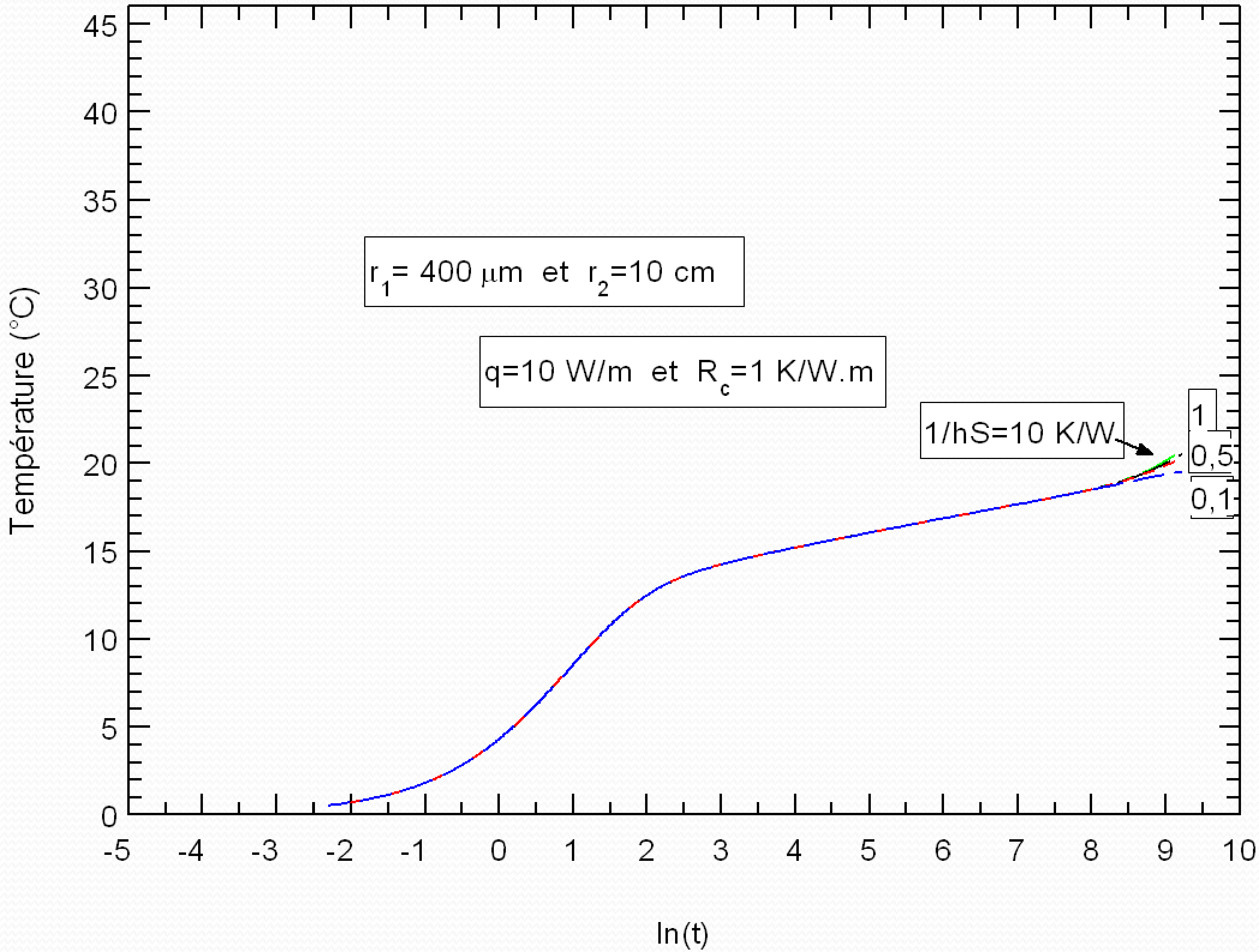
Effect of the Hot-Wire *Capacity*



Effect of the *Finite* Medium / *Heat Losses (r_2=3 cm)*



Effect of the *Finite* Medium / *Heat Losses* (r_2=10 cm)



Sensitivity Curves to the Parameters

Parameters :

- –*"Hot-Wire"* Thermal Conductivity
- –*"Hot-Wire"* Thermal Diffusivity
- –*"Medium"* Thermal Conductivity
- –*"Medium"* Thermal Diffusivity
- – Contact Resistance *"Hot-Wire / Medium"*
- – *"Convective"* resistance (Heat Losses)

- Sensitivities :

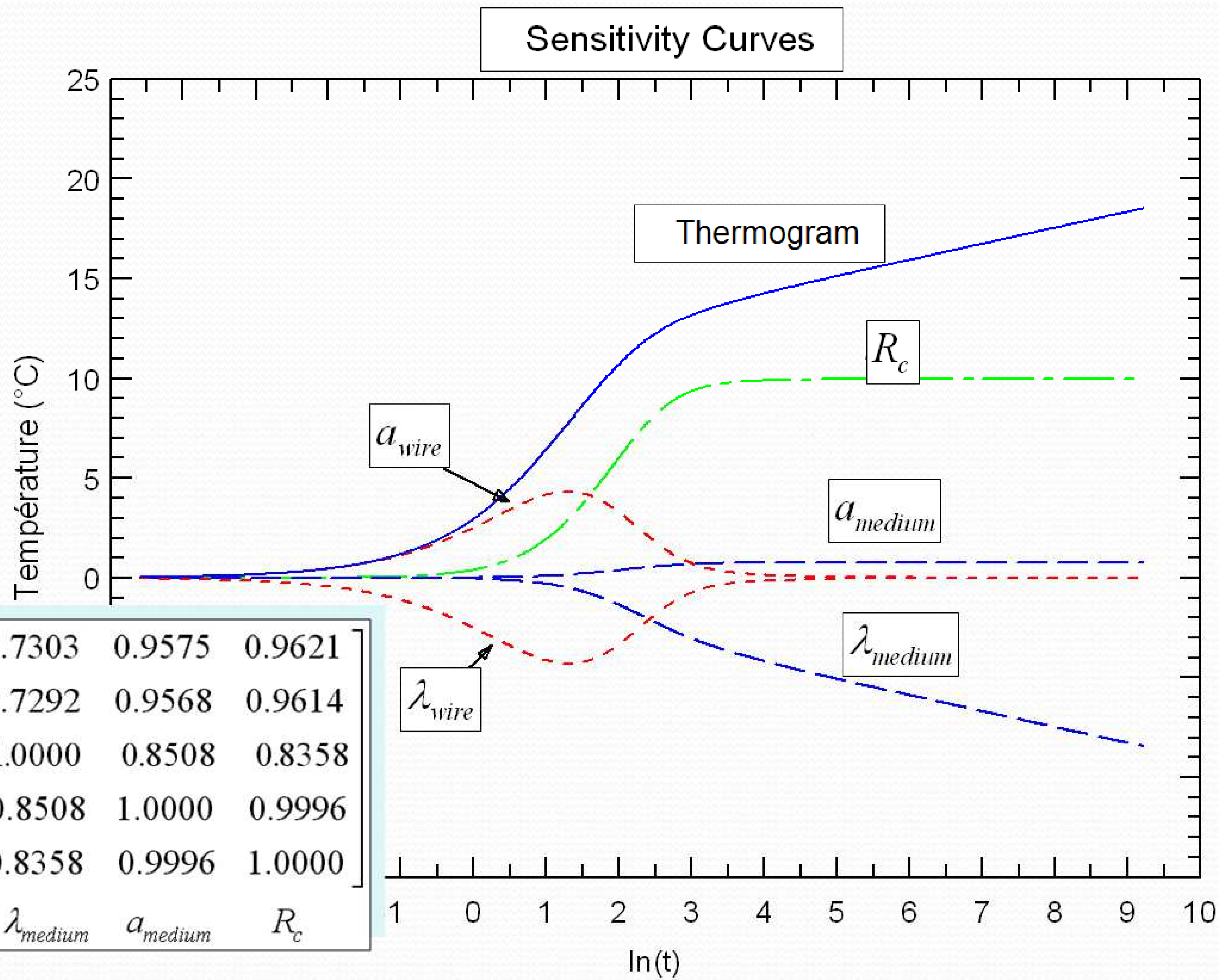
$$X_i(t,\mathbf{K}) = \frac{\partial \mathrm{F}(t,\mathbf{K})}{\partial K_i} \quad \text{et} \quad X_i^*(t) = K_i X_i(t) = K_i \frac{\partial \mathrm{F}(t,\mathbf{K})}{\partial K_i}$$

« reduced »

- Correlation Factor :

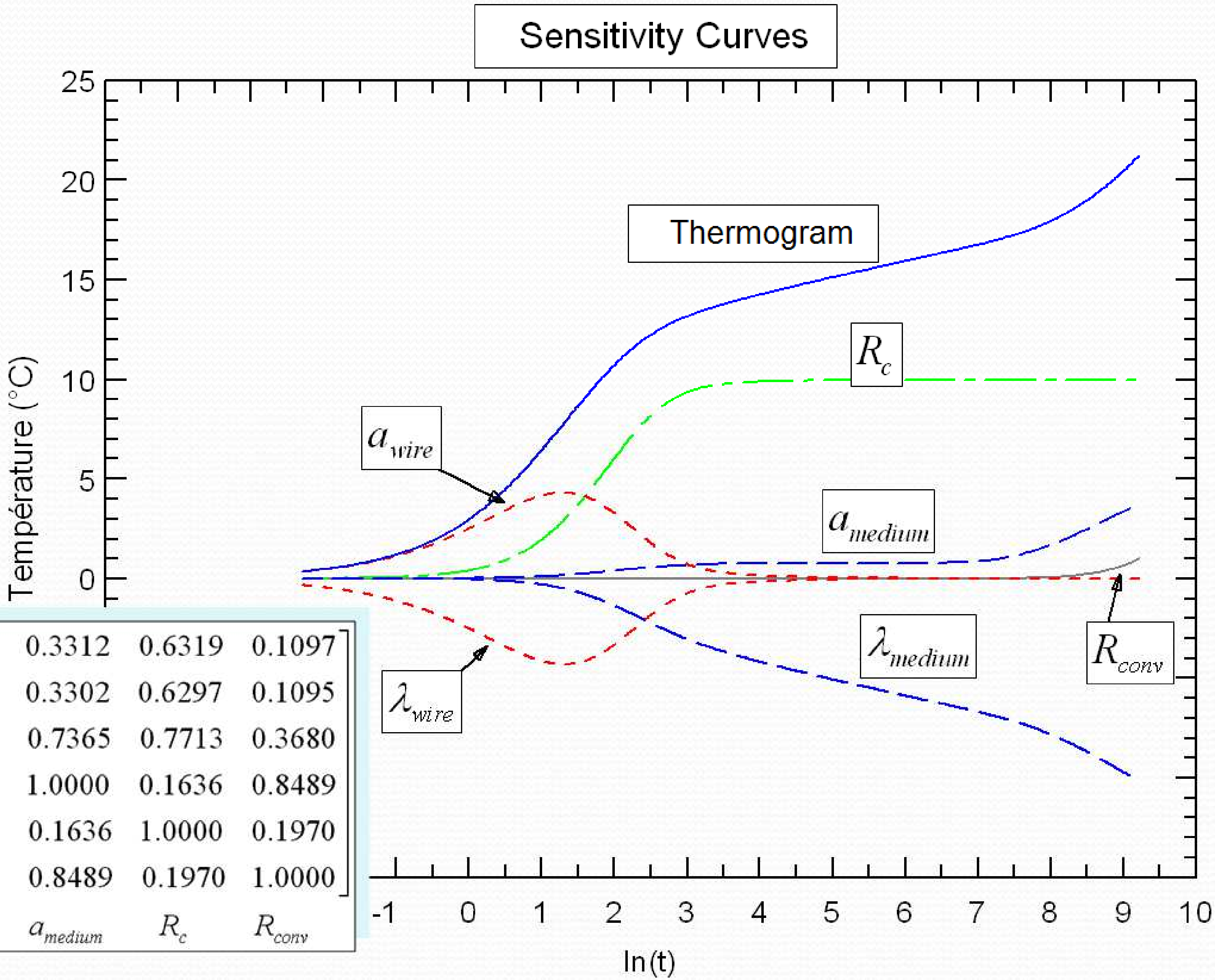
$$\rho(K_i, K_j) = \frac{\mathrm{Cov}(K_i, K_j)}{\sqrt{\mathrm{Var}(K_i) \cdot \mathrm{Var}(K_j)}}$$

Semi-Infinite Medium



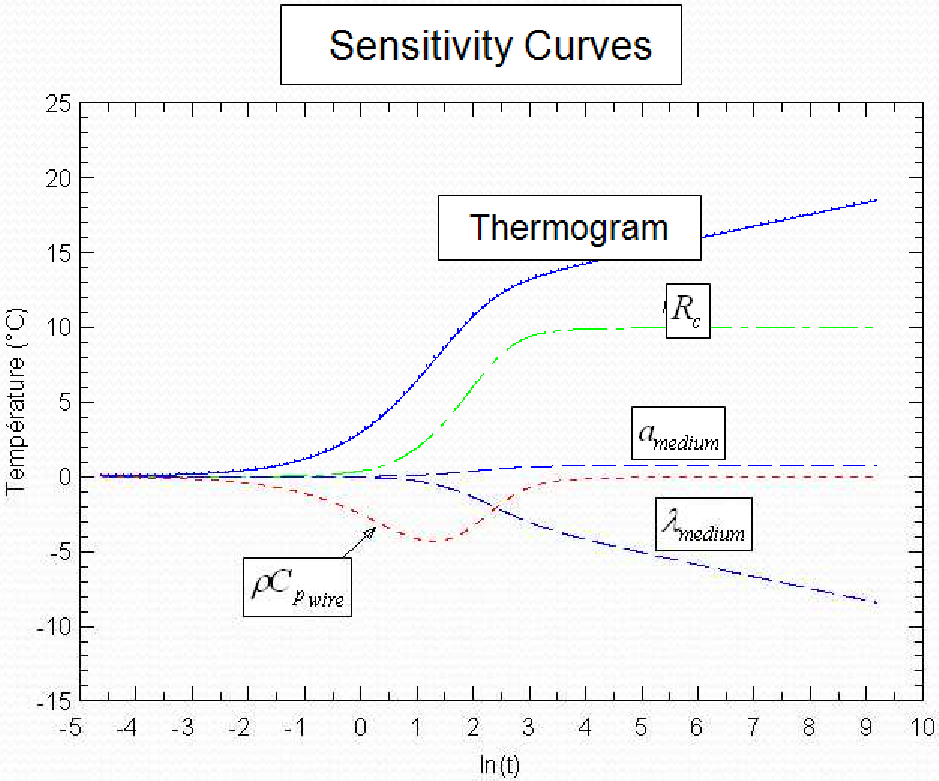
	λ_{wire}	a_{wire}	λ_{medium}	a_{medium}	R_c
λ_{wire}	1.0000	1.0000	0.7303	0.9575	0.9621
a_{wire}	1.0000	1.0000	0.7292	0.9568	0.9614
λ_{medium}	0.7303	0.7292	1.0000	0.8508	0.8358
a_{medium}	0.9575	0.9568	0.8508	1.0000	0.9996
R_c	0.9621	0.9614	0.8358	0.9996	1.0000

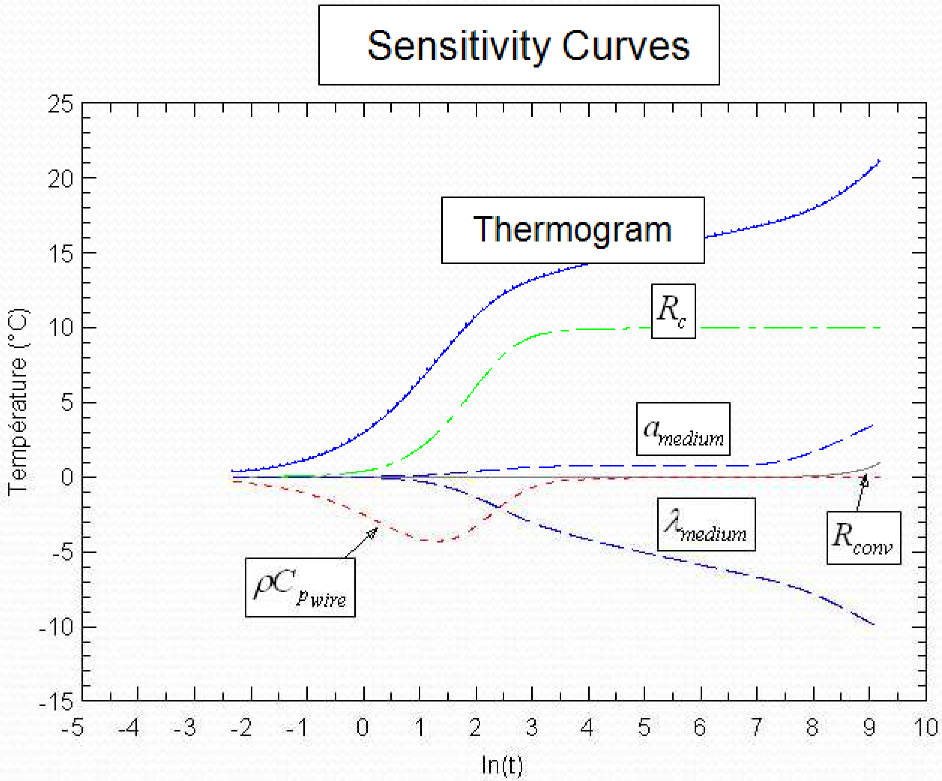
Finite Medium



	λ_{wire}	a_{wire}	λ_{medium}	a_{medium}	R_c	R_{conv}
λ_{wire}	1.0000	1.0000	0.6110	0.3312	0.6319	0.1097
a_{wire}	1.0000	1.0000	0.6089	0.3302	0.6297	0.1095
λ_{medium}	0.6110	0.6089	1.0000	0.7365	0.7713	0.3680
a_{medium}	0.3312	0.3302	0.7365	1.0000	0.1636	0.8489
R_c	0.6319	0.6297	0.7713	0.1636	1.0000	0.1970
R_{conv}	0.1097	0.1095	0.3680	0.8489	0.1970	1.0000

Model Reduction





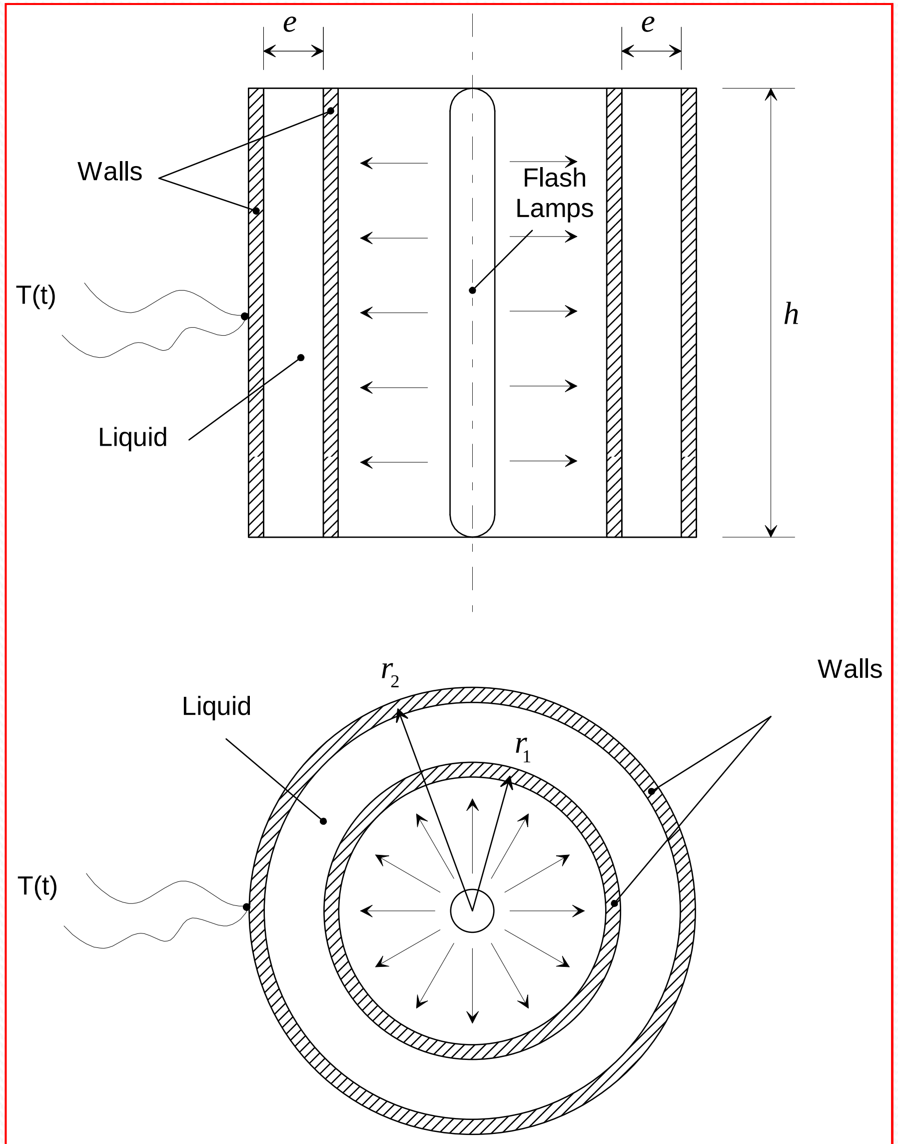
	$(\rho C_p)_{wire}$	λ_{medium}	a_{medium}	R_c
$(\rho C_p)_{wire}$	1.0000	0.4754	0.8277	0.8477
λ_{medium}	0.4754	1.0000	0.7688	0.7145
a_{medium}	0.8277	0.7688	1.0000	0.9965
R_c	0.8477	0.7145	0.9965	1.0000

	$(\rho C_p)_{wire}$	λ_{medium}	a_{medium}	R_c	R_{conv}
$(\rho C_p)_{wire}$	1.0000	0.4527	0.1884	0.4853	0.0379
λ_{medium}	0.4527	1.0000	0.7152	0.6287	0.3826
a_{medium}	0.1884	0.7152	1.0000	0.0612	0.8664
R_c	0.4853	0.6287	0.0612	1.0000	0.3448
R_{conv}	0.0379	0.3826	0.8664	0.3448	1.0000

Estimations with Models Without Degrees of Freedom

Case of the Liquid Flash Experiment

Introduction



☞ **Good contact between liquid and walls**

☞ **One-dimensional Heat Transfer**

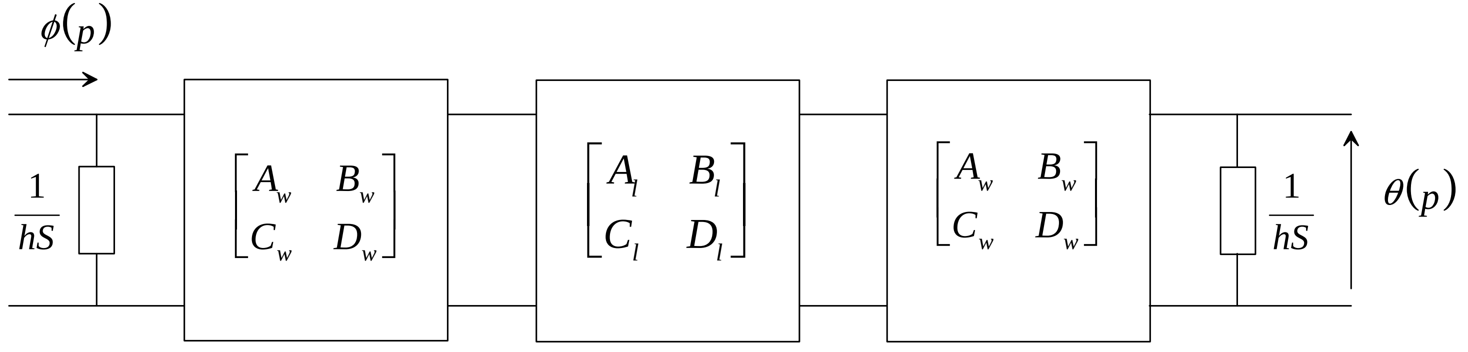
☞ **Presence of the Natural Convection requires to work in a "pseudo-conduction" regime**

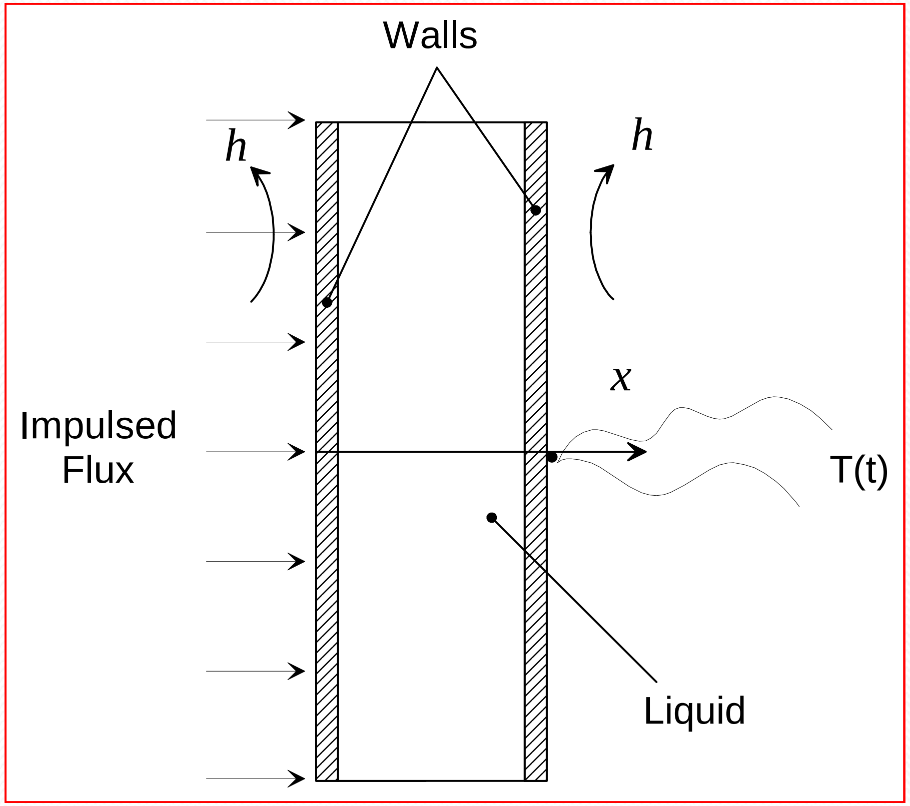
(choose the aspect ratio of the measurement cell e/h<<1)

Principle of the Measurement

Model : Problem in Pure Conduction

Quadrupole Representation





$1/hS$ *: Convective Heat Losses*

$$A_i = D_i = \cosh\left(\sqrt{\frac{pe_i^2}{a_i}}\right) \quad , \quad B_i = \frac{1}{\lambda_i S\sqrt{\frac{p}{a_i}}}\sinh\left(\sqrt{\frac{pe_i^2}{a_i}}\right)$$

and $$C_i = \lambda_i S\sqrt{\frac{p}{a_i}}\sinh\left(\sqrt{\frac{pe_i^2}{a_i}}\right)$$

e_i *: thickness of the material*

a_i *: thermal diffusivity*

λ_i *: thermal conductivity*

Solution

The rear-face temperature θ(p) is given by:

$$\theta(p)=\frac{\phi(p)}{\mathscr{C}+2\mathscr{A}hS+\mathscr{B}(hs)^2}$$

$\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ represent the coefficients of the transfer matrix:

$$\begin{bmatrix}\mathscr{A} & \mathscr{B}\\ \mathscr{C} & \mathscr{D}\end{bmatrix}=\begin{bmatrix}A_w & B_w\\ C_w & A_w\end{bmatrix}\begin{bmatrix}A_l & B_l\\ C_l & A_l\end{bmatrix}\begin{bmatrix}A_w & B_w\\ C_w & A_w\end{bmatrix}$$

With:

$$\mathscr{A}=(A_wA_l+B_wC_l)A_w+(A_wB_l+B_wA_l)C_w$$

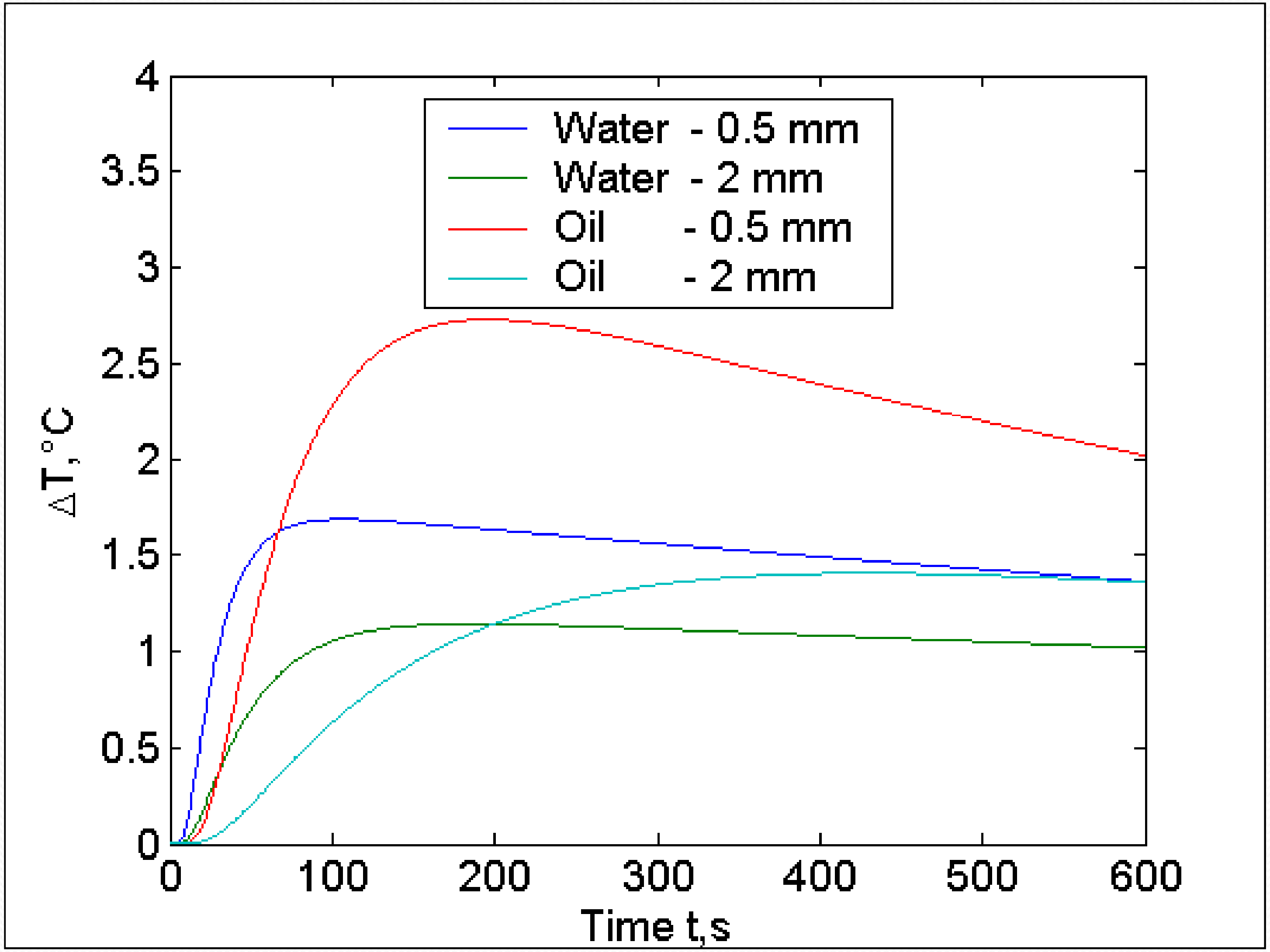
$$\mathscr{B}=(A_wA_l+B_wC_l)B_w+(A_wB_l+B_wA_l)A_w$$

$$\mathscr{C}=(C_wA_l+A_wC_l)A_w+(C_wB_l+A_wA_l)C_w$$

For a Heat Pulse (Dirac of Flux) $\rightarrow$ $\phi(p)=Q$ ⇨ $T(t)=\mathscr{L}^{-1}\theta(p)$

Simulation Examples

- $e_l = 4{,}5\,mm$
  $h = 5\,W.m^{-2}.K^{-1}$
- **Water:**
  $\lambda_l = 0{,}597\,W.m^{-1}.K^{-1}$
  $a_l = 1{,}43.10^{-7}\,m^2.s^{-1}$
- **Oil:**
  $\lambda_l = 0{,}132\,W.m^{-1}.K^{-1}$
  $a_l = 7{,}33.10^{-8}\,m^2.s^{-1}$
- **Walls (copper):**
  $\lambda_w = 395\,W.m^{-1}.K^{-1}$
  $a_w = 1{,}15.10^{-4}\,m^2.s^{-1}$
  $e_w = 0{,}5\ or\ 2\,mm$
- $Q/S = 4.10^4\ J.m^{-2}$



Generalities

4 Unknown Parameters:

$$\beta_1 = \frac{e_l}{\sqrt{a_l}}, \quad \beta_2 = \frac{e_l}{\lambda_l}, \quad \beta_3 = \frac{Q}{S} \quad and \quad \beta_4 = h$$

$$\Rightarrow \quad T = f\left(t, \beta_1, \beta_2, \beta_3, \beta_4\right) = f\left(t, \beta\right)$$

$\Rightarrow$ *Reduced Sensitivity Coefficient:*

$$X_j^{\ *}\left(t, \beta\right) = \beta_j \frac{\partial T}{\partial \beta_j}\left(t, \beta\right)$$

- X_j^* maximum $\rightarrow$ small error
- X_j^* proportional $\rightarrow$ parameters are correlated

Stochastical Approach

$$S(\beta) = \sum_{i=1}^{n} (Y_i - T(t_i,\beta))^2 \rightarrow \quad \frac{\partial S(\beta)}{\partial \beta} = 0 = \sum_{i=1}^{n} \frac{\partial T(t_i,\beta)}{\partial \beta_j} (Y_i - T(t_i,\beta)) = 0 \ (\vee \beta_j)$$

$\varepsilon(t)$ *being the noise at time t*

$$\hat{\beta}^{(n+1)} = \beta^{(n)} + \left(X^{(n)^t} X^{(n)}\right)^{-1} X^{(n)^t} \varepsilon(t)$$

$$E(\hat{\beta}) = \beta$$

: *expected values of parameters (unbiased estimator)*

$$V(\hat{\beta}) = \sigma_n^{\ 2} (X^t X)^{-1} = \sigma_n^{\ 2} \begin{bmatrix} Var(\beta_i) & Cov(\beta_i, \beta_j) \\ Cov(\beta_i, \beta_j) & Var(\beta_j) \end{bmatrix}$$

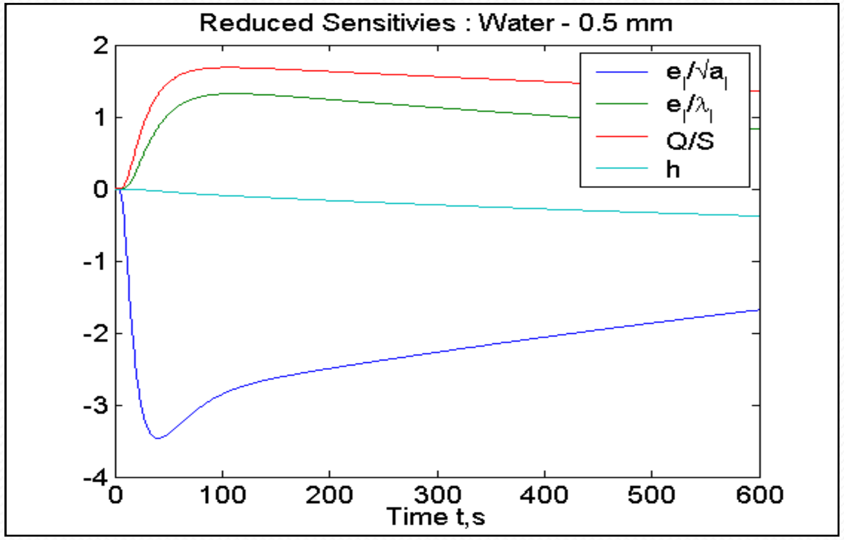
$$Y_i = T(t_i, \beta) + \varepsilon_i$$

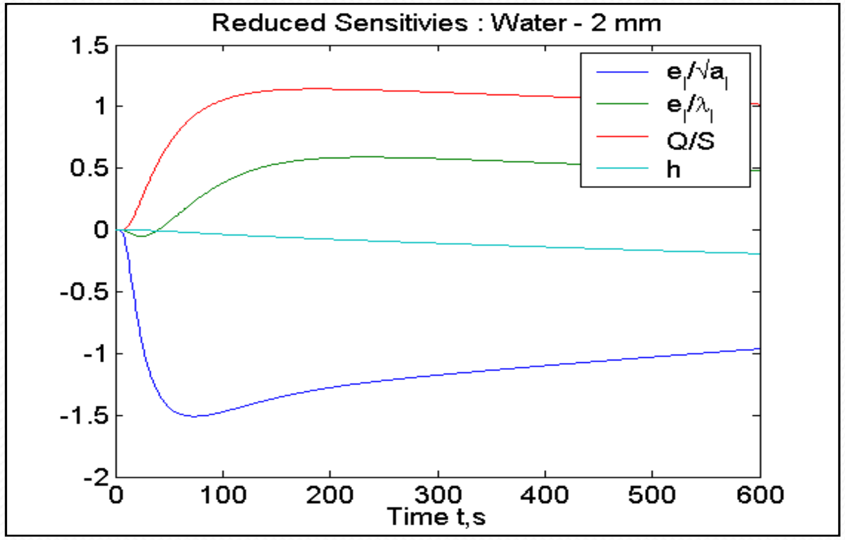
: *covariance matrix (σ_n : standard deviation of noise)*

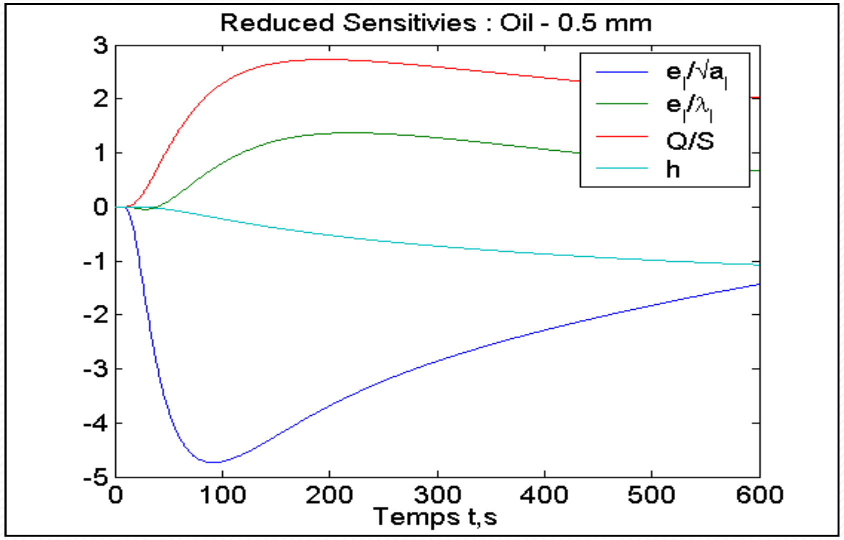
$$\rho(\beta_i, \beta_j) = \frac{\mathrm{Cov}(\beta_i, \beta_j)}{\sqrt{\mathrm{Var}(\beta_i) \cdot \mathrm{Var}(\beta_j)}}$$

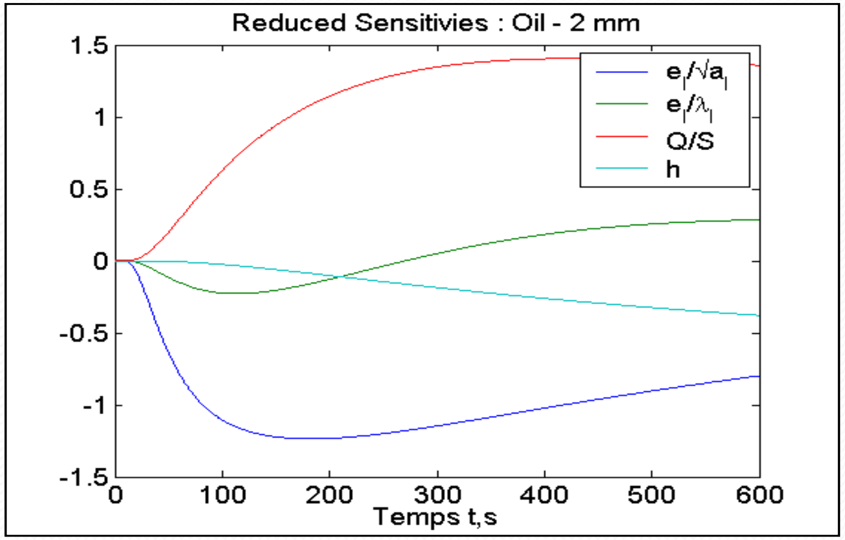
- ***Good Estimation*** *if*
  - *Large Sensitivities*
  - *Correlation coefficients are far from unity*
  - *Variances are small*

Sensitivity Analysis









*The estimation problem is **non-linear** → the estimation depends on the nominal values of the parameters* → **an optimal walls thickness exists**

Variance-Covariance Matrix

Water – 0,5 mm	Water – 2 mm
0.3394 -2.3464 2.4913 1.4724 -2.3464 16.5302 -17.4179 -9.4267 2.4913 -17.4179 18.4144 10.4120 1.4724 -9.4267 10.4120 9.7216	0,3218 -0,8419 0,7528 -0,5216 -0,8419 2,4531 -2,0146 2,5528 0,7528 -2,0146 1,7770 -1,3092 -0,5216 2,5528 -1,3092 8,7357
Oil – 0,5 mm	**Oil – 2 mm**
0,0649 -0,2870 0,2533 0,1216 -0,2870 1,3529 -1,1408 -0,4388 0,2533 -1,1408 0,9958 0,4599 0,1216 -0,4388 0,4599 0,3979	0,1920 -0,4540 0,1500 -0,2349 -0,4540 1,3544 -0,2825 1,0794 0,1500 -0,2825 0,1413 -0,0219 -0,2349 1,0794 -0,0219 1,4113

Variance-Covariance Matrix

Parameters Substitution

New Parameters: $\beta_1 = \frac{e_l}{\sqrt{a_l}}$, $\beta_2 = \rho c_l e_l$, $\beta_3 = \frac{Q}{S}$ and $\beta_4 = h$

Water – 1 mm

4 parameters: β_1, β_2, β_3 and β_4

Covariance

0.2567	1.5697	1.0776	0.0993
1.5697	9.8171	6.6809	-0.2249
1.0776	6.6809	4.5673	0.1590
0.0993	-0.2249	0.1590	4.9007

Correlation

1.0000	0.9888	0.9952	0.0886
0.9888	1.0000	0.9977	-0.0324
0.9952	0.9977	1.0000	0.0336
0.0886	-0.0324	0.0336	1.0000

3 parameters (β_2 fixed): β_1, β_3 and β_4

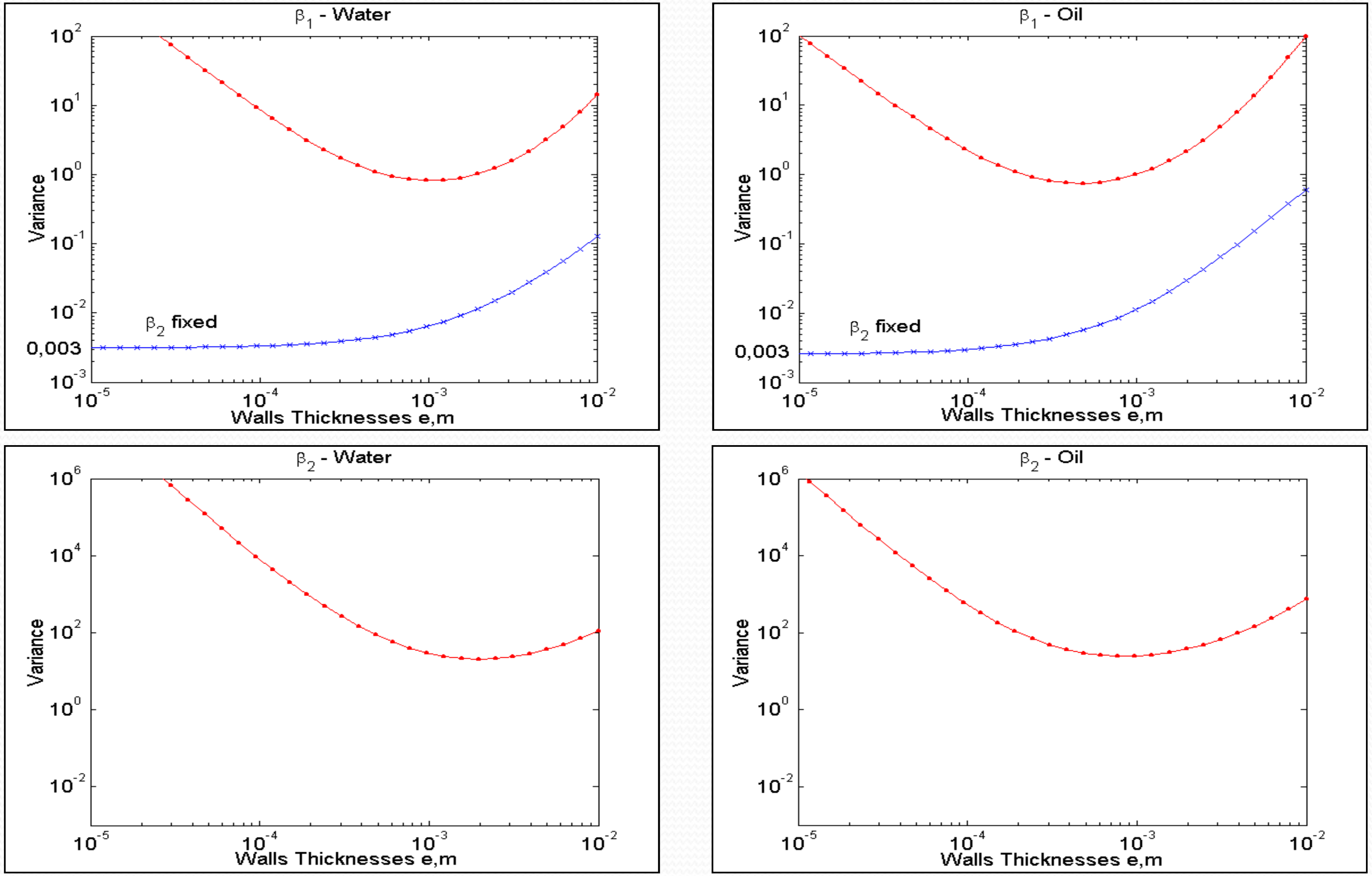
Covariance

0.0057	0.0094	0.1353
0.0094	0.0208	0.3121
0.1353	0.3121	4.8955

Correlation

1.0000	0.8596	0.8074
0.8596	1.0000	0.9777
0.8074	0.9777	1.0000

Optimization of the walls thicknesses



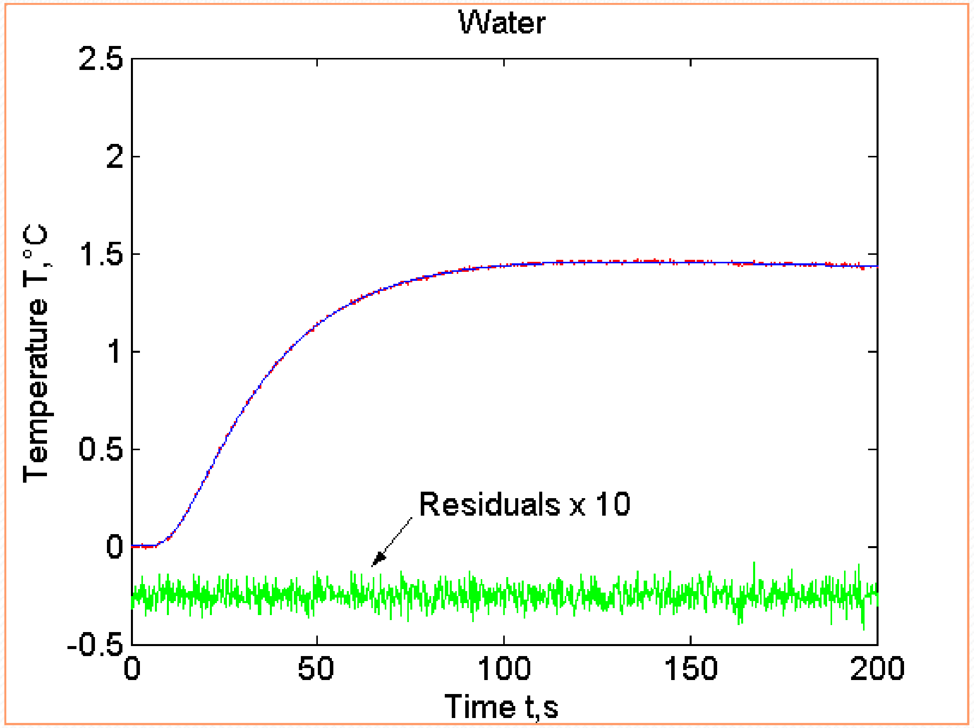
- Fluid (Water) [e_f =4,5 mm, λ_f=0,597 $W.m^{-1}.K^{-1}$, a_l=1,43.10^{-7} $m^2.s^{-1}$, ρc_l=4,17.10^6 $J.m^{-3}.K^{-1}$
- Fluid (Oil) [e_f =4,5 mm, λ_f=0,132 $W.m^{-1}.K^{-1}$, a_l=7,33.10^{-7} $m^2.s^{-1}$, ρc_l=1,8.10^6 $J.m^{-3}.K^{-1}$]
- Walls (Copper) [λ_w=395 $W.m^{-1}.K^{-1}$, a_w=1,15.10^{-4} $m^2.s^{-1}$, ρc_w=3,43.10^6 $J.m^{-3}.K^{-1}$]
- h=5 $W.m^{-2}.K^{-1}$ – Q/S=4.10^4 $J.m^{-2}$

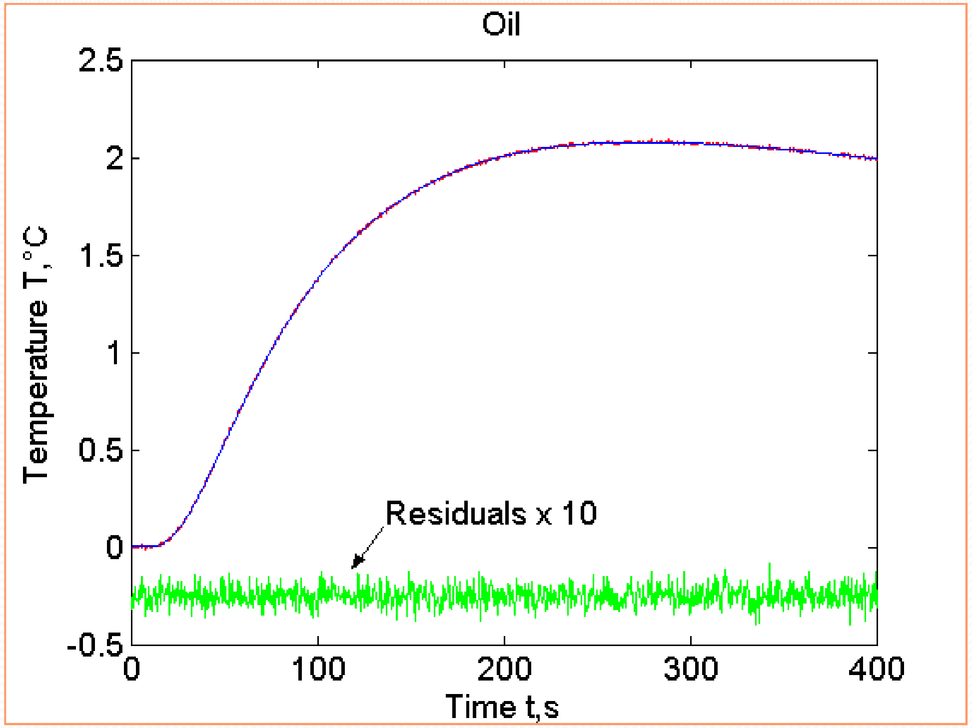
Inverse Method on Simulated Thermograms

- *Estimation Program:* Levenberg-Marquardt Algorithm with 4 parameters

$$\beta_1 = e_l / \sqrt{a_l}\,, \quad \beta_2 = \rho c_l e_l\,, \quad \beta_3 = Q/S \quad and \quad \beta_4 = h$$

- *Standard deviation of the noise:* σ_n= 0.005 K





- Fluid (Water) [e_f =4,5 mm, λ_f=0,597 $W.m^{-1}.K^{-1}$, a_l=1,43.10^{-7} $m^2.s^{-1}$, ρc_l=4,17.10^6 $J.m^{-3}.K^{-1}$
- Fluid (Oil) [e_f =4,5 mm, λ_f=0,132 $W.m^{-1}.K^{-1}$, a_l=7,33.10^{-7} $m^2.s^{-1}$, ρc_l=1,8.10^6 $J.m^{-3}.K^{-1}$]
- Walls (Copper) [λ_w=395 $W.m^{-1}.K^{-1}$, a_w=1,15.10^{-4} $m^2.s^{-1}$, ρc_w=3,43.10^6 $J.m^{-3}.K^{-1}$]
- h=5 $W.m^{-2}.K^{-1}$ – Q/S=4.10^4 $J.m^{-2}$

Estimated Values : 4 parameters

4 parameters: $e_l/\sqrt{a_l}$, $\rho c_l e_l$, Q/S and h

Water		Oil	
Parameters		**Parameters**	
Nominal	*Estimated*	**Nominal**	*Estimated*
a_l=1,43.10^{-7} $m^2.s^{-1}$	a_l=1,417.10-7 $m^2.s^{-1}$	a_l=7,33.10^{-7} $m^2.s^{-1}$	a_l=7,284.10^{-7} $m^2.s^{-1}$
ρc_l=4,17.10^6 $J.m^{-3}.K^{-1}$	ρc_l=4,276.10^6 $J.m^{-3}.K^{-1}$	ρc_l=1,8.10^6 $J.m^{-3}.K^{-1}$	ρc_l=1,827.10^6 $J.m^{-3}.K^{-1}$
h=5 $W.m^{-2}.K^{-1}$	h=5,083 $W.m^{-2}.K^{-1}$	h=5 $W.m^{-2}.K^{-1}$	h=5,014 $W.m^{-2}.K^{-1}$
Q/S=4.10^4 $J.m^{-2}$	Q/S=4,071.10^4 $J.m^{-2}$	Q/S=4.10^4 $J.m^{-2}$	Q/S=4,026.10^4 $J.m^{-2}$

Covariance

Water				Oil			
0.2604	1.6099	1.1144	0.1305	0.0885	0.4597	0.1814	-0.0129
1.6099	10.1769	6.9852	-0.0272	0.4597	2.5109	0.9409	-0.2640
1.1144	6.9852	4.8152	0.2932	0.1814	0.9409	0.3747	-0.0099
0.1305	-0.0272	0.2932	4.8916	-0.0129	-0.2640	-0.0099	0.3976

Correlation

Water				Oil			
1.0000	0.9890	0.9952	0.1156	1.0000	0.9753	0.9962	-0.0686
0.9890	1.0000	0.9978	-0.0038	0.9753	1.0000	0.9700	-0.2642
0.9952	0.9978	1.0000	0.0604	0.9962	0.9700	1.0000	-0.0257
0.1156	-0.0038	0.0604	1.0000	-0.0686	-0.2642	-0.0257	1.0000

$\sigma_a = 0,5\%$ $\sigma_{\rho c} = 1,6\%$ $\sigma_a = 0,3\%$ $\sigma_{\rho c} = 0,8\%$

Estimated Values: 3 parameters

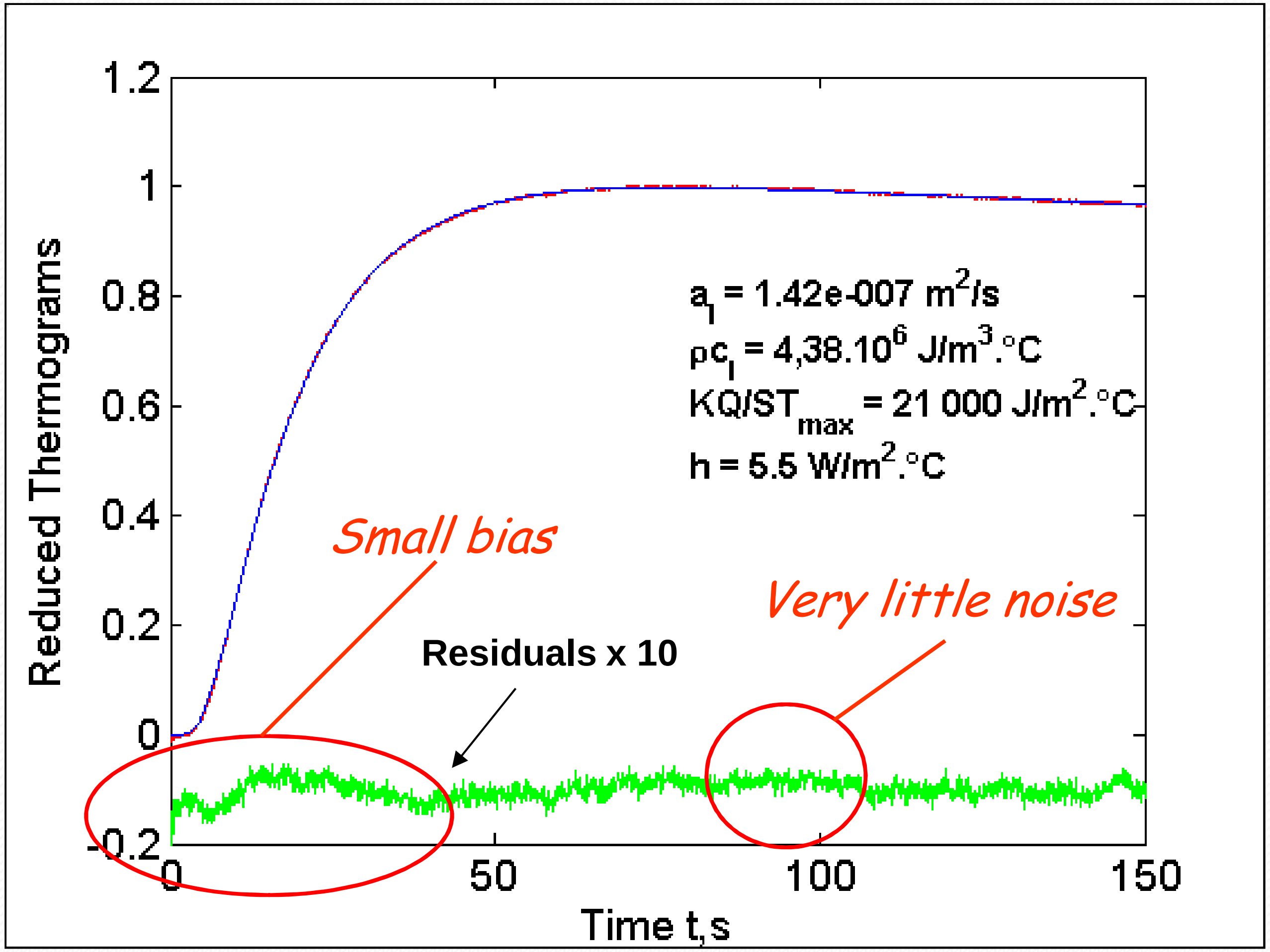
4 parameters ($\rho c_l e_l$ fixed): $e_l/\sqrt{a_l}$, Q/S and h			
Water		Oil	
Parameters (ρc_l=4,17.10^6 J.m^{-3}.K^{-1})		Parameters (ρc_l=1,8.10^6 J.m^{-3}.K^{-1})	
Nominal	*Estimated*	**Nominal**	*Estimated*
a_l=1,43.10^{-7} m^2.s^{-1} h=5 W.m^{-2}.K^{-1} Q/S=4.10^4 J.m^{-2}	a_l=1,428.10-7 m^2.s^{-1} h=5,084 W.m^{-2}.K^{-1} Q/S=4,005.10^4 J.m^{-2}	a_l=7,33.10^{-7} m^2.s^{-1} h=5 W.m^{-2}.K^{-1} Q/S=4.10^4 J.m^{-2}	a_l=7,323.10^{-7} m^2.s^{-1} h=5,022 W.m^{-2}.K^{-1} Q/S=4,005.10^4 J.m^{-2}

Covariance (Water)			Covariance (Oil)		
0.0058	0.0095	0.1347	0.0044	0.0092	0.0354
0.0095	0.0211	0.3117	0.0092	0.0223	0.0888
0.1347	0.3117	4.8196	0.0354	0.0888	0.3651

Correlation (Water)			Correlation (Oil)		
1.0000	0.8609	0.8089	1.0000	0.9339	0.8879
0.8609	1.0000	0.9779	0.9339	1.0000	0.9840
0.8089	0.9779	1.0000	0.8879	0.9840	1.0000

$\sigma_a = 0{,}08\%$ $\sigma_a = 0{,}06\%$

Estimated Values: 4 parameters



4 parameters: $e_l/\sqrt{a_l}$, $\rho c_l e_l$, Q/s and h

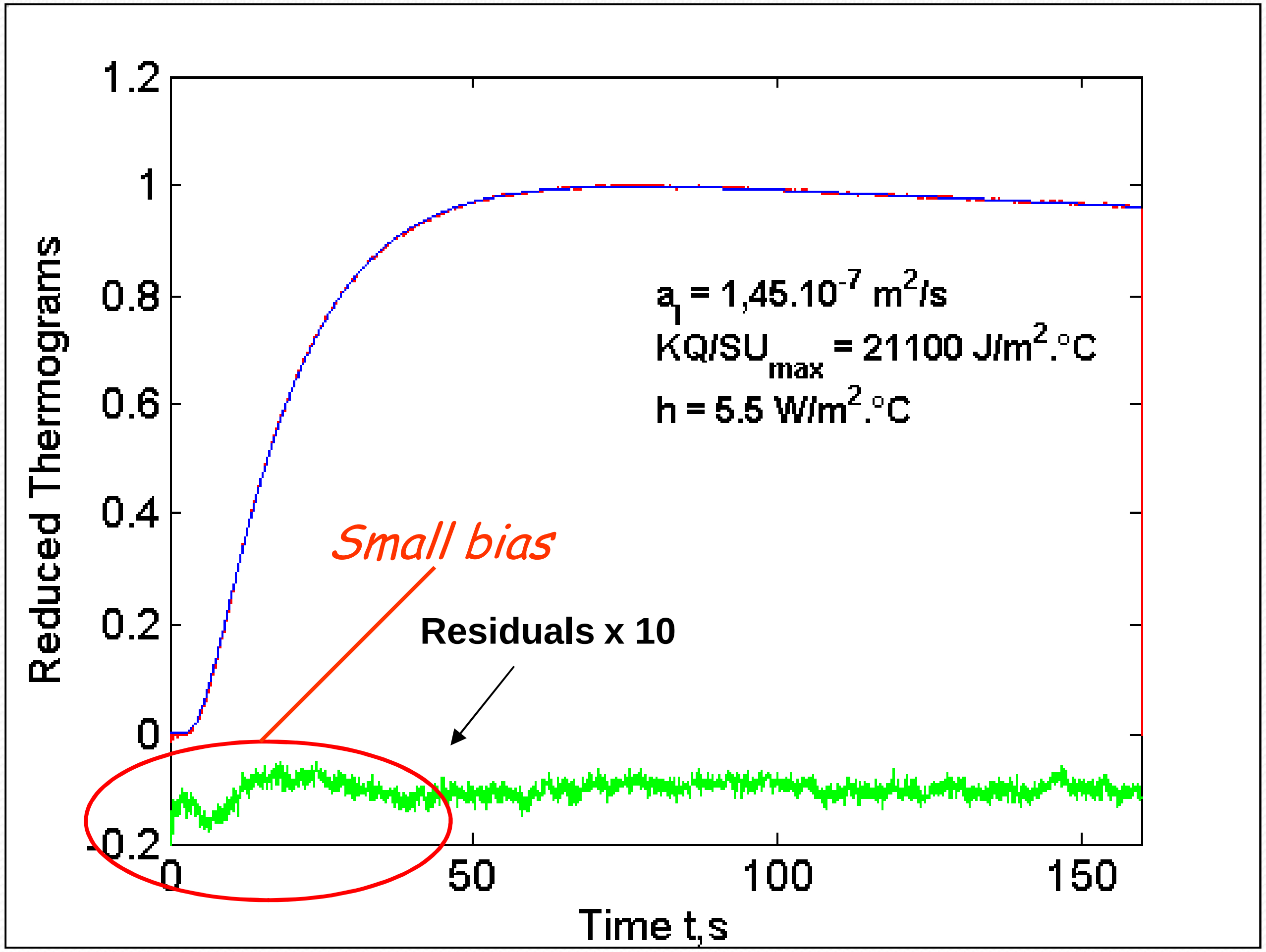
Covariance

0.1453	0.6414	0.3682	0.0941
0.6414	2.9094	1.6528	0.1704
0.3682	1.6528	0.9452	0.1949
0.0941	0.1704	0.1949	1.6610

$$\begin{cases} a = 1{,}42 \pm 0{,}025\,.10^{-7}\ m^2/s \\ \rho c = 4{,}38 \pm 0{,}170\,.10^{6}\ J/m^3.K \end{cases}$$

❖ *Estimation on an Experimental Thermogram (Water) – 4 Parameters Model*

Estimated Values: 4 parameters



$\rho c_l e_l$ is fixed to **4,15.10^6 J/m^3°C**

3 parameters: $e_l/\sqrt{a_l}$, Q/s and h

Covariance

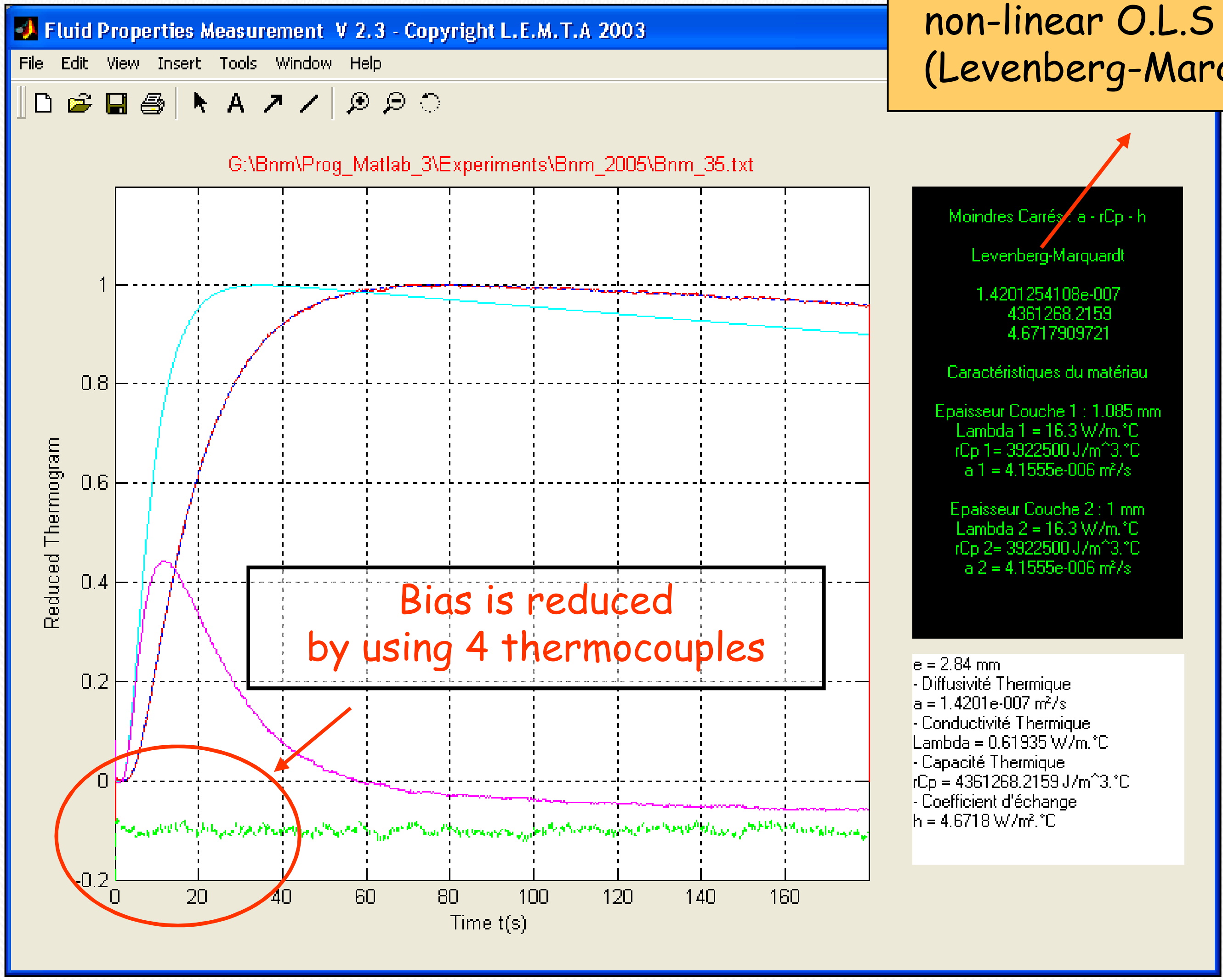
0.0039	0.0039	0.0557
0.0039	0.0064	0.0959
0.0557	0.0959	1.5606

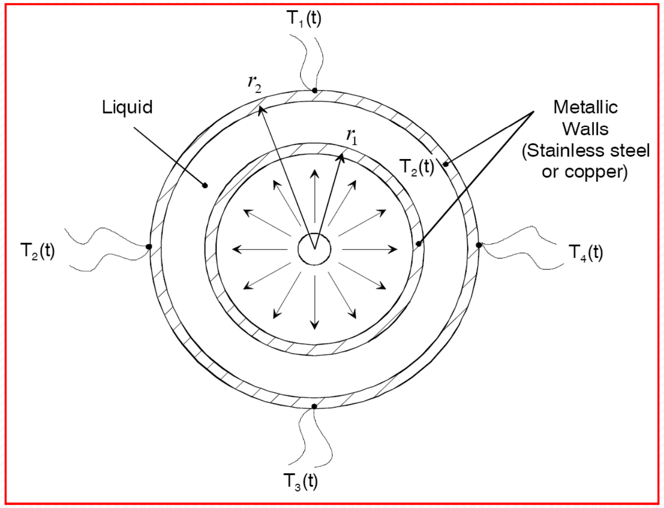
$$a = 1{,}45 \pm 0{,}004 \,.10^{-7}\; m^2/s$$

❖ *Estimation on an Experimental Thermogram (Water) – 3 Parameters Model*

Estimated Values: 4 parameters

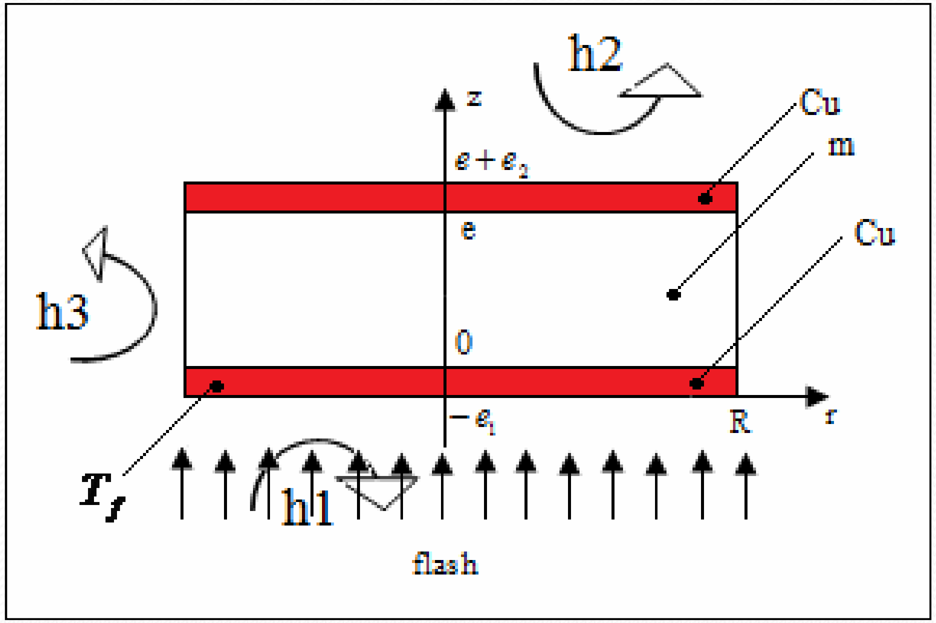
Estimation by a
non-linear O.L.S method
(Levenberg-Marquardt)

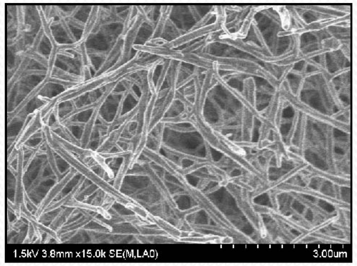


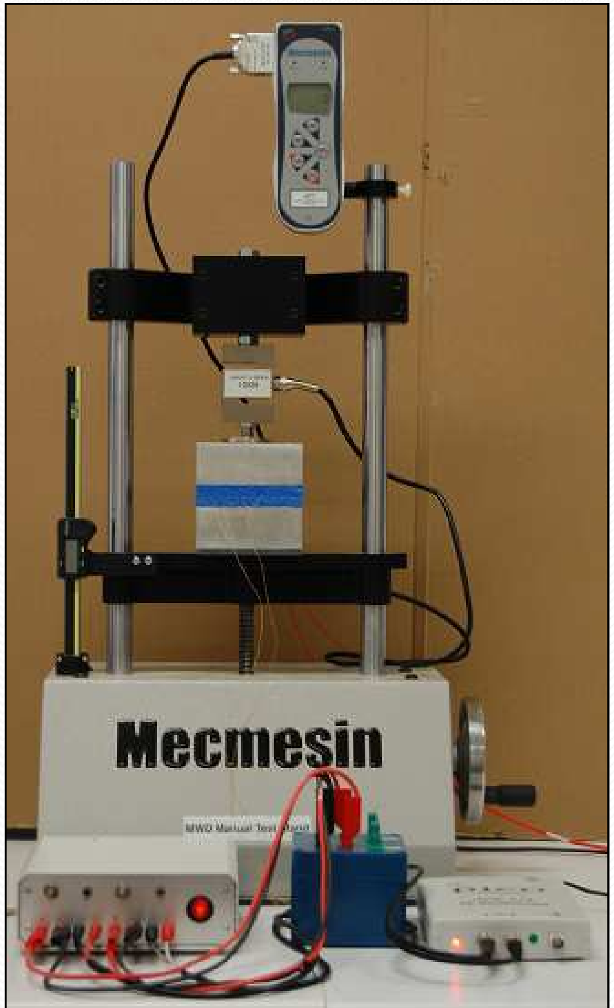


Thermal Charaterization of Aerogels

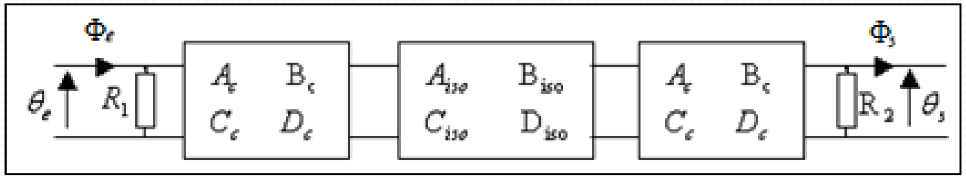
Low Molecular Weight Aerogels (High insulating material)





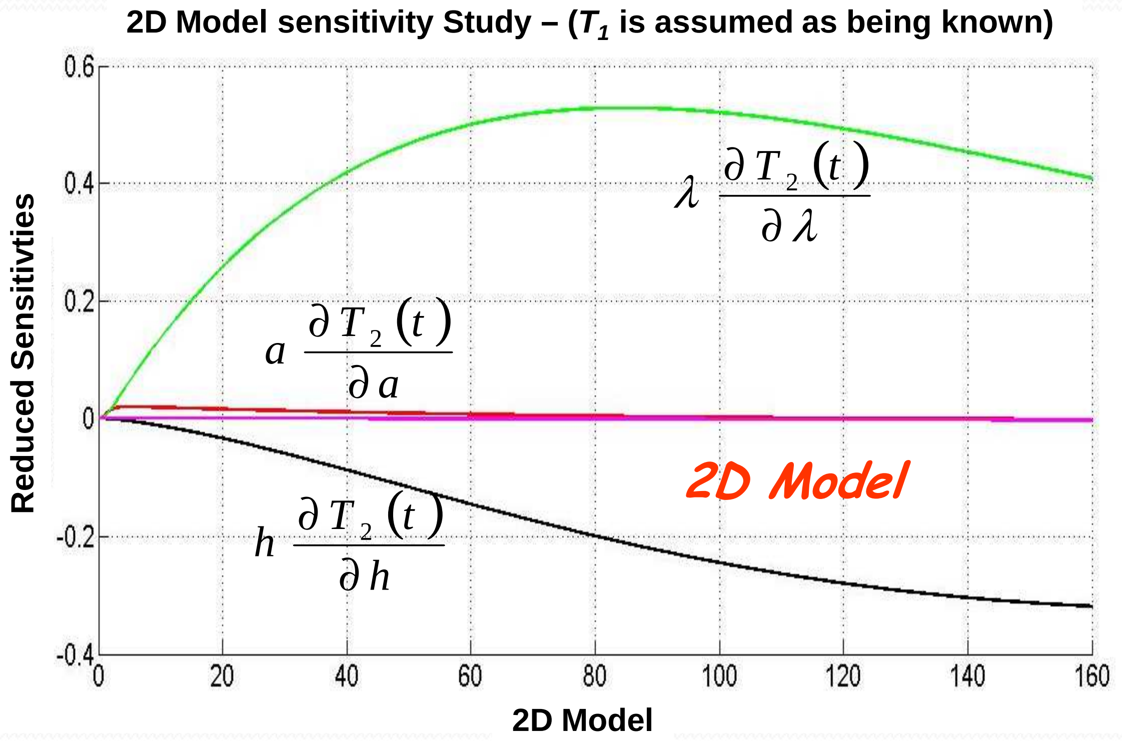


Principle of the experiment



Sensitivity Curves

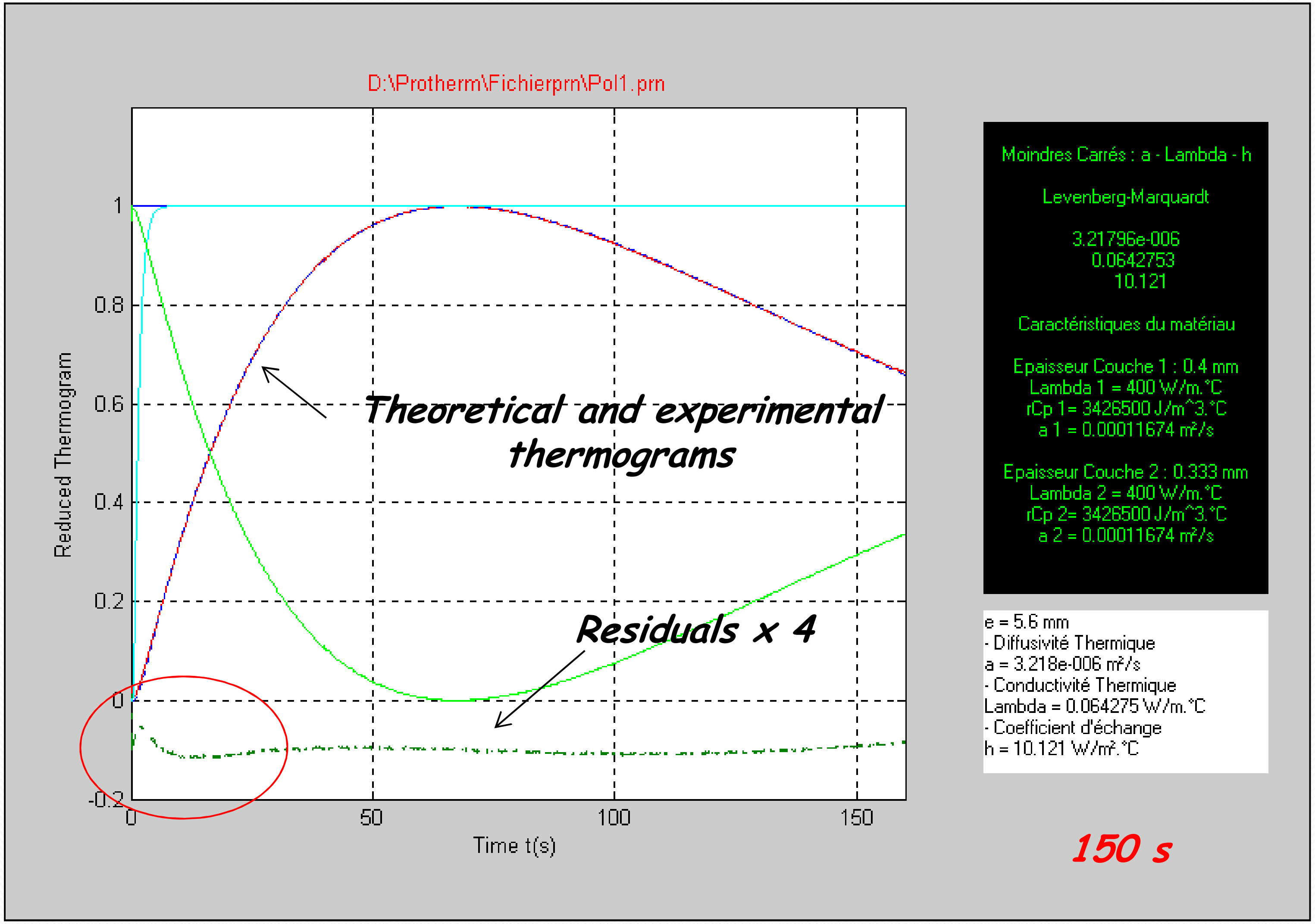
High Insulating Material : λ =0,02 $W.m^{-1}.K^{-1}$, ρc = 5000 $J.m^{-3}.K^{-1}$, a= 4.10^{-6} $m^2.s^{-1}$



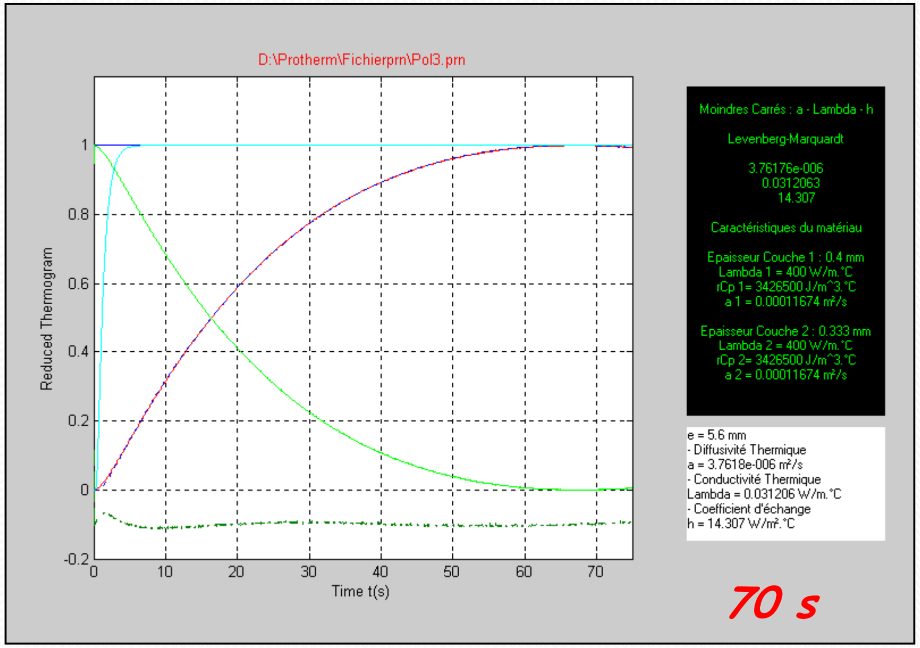
- High sensitivity to conductivity λ , low sensitivity to diffusivity *a*
- Conductivity λ is non-correlated with heat losses *h* and diffusivity *a*

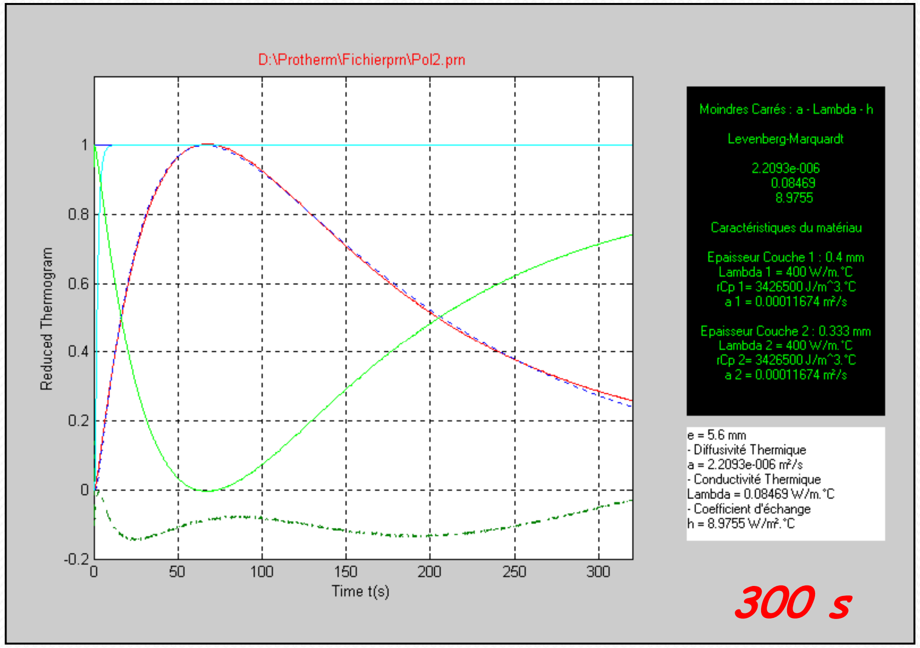
High Accurate Estimation of λ is <u>theoretically</u> possible

Low Weight Insulating Material



Low Weight Insulating Material





Time Interval	70 s	150 s	300 s
a (m²/s)	$3.76.10^{-6}$	$3.22.10^{-6}$	$2.21.10^{-6}$
λ (W/m. C)	0.031	0.064	0.084

Rigid Foam* : *a=4,68 to 4,54.10^{-7} m²/s* and *λ=0,039 to 0,042 W/m. C

Taking into account the Bias to reduce the variances on estimated parameters

Case of the classical Flash Method

I. Parameter Estimation by taking into account the Bias

We have:

$$\mathbf{F}\left(t,\hat{\beta}_r,\beta_{c_{nom}}\right) = \mathbf{F}\left(t,\beta_r,\beta_c\right) + \mathbf{X}_r\big|_{\beta}\left(\hat{\beta}_r - \beta_r\right) + \mathbf{X}_c\big|_{b}\left(\beta_{c_{nom}} - \beta_c\right)$$

$$\tilde{\mathbf{F}}\left(t,\hat{\beta}_r,\beta_{c_{nom}}\right) = \mathbf{F}\left(t,\beta_r,\beta_c\right) + \left(\mathbf{X}_r \middle| \mathbf{X}_c\right)\begin{pmatrix} b_{\beta_r} = \hat{\beta}_r - \beta_r \\ e_{b_c} = \beta_{c_{nom}} - \beta_c \end{pmatrix} \qquad (1)$$

With:

$b_{\beta_r} = \hat{\beta}_r - \beta_r$: « bias on estimated parameters »

$e_{\beta_c} = \beta_{c_{nom}} - \beta_c$: « error on fixed parameters »

$\mathbf{F}\left(t,\beta_r,\beta_c\right)$: Detailed model

$\tilde{\mathbf{F}}\left(t,\hat{\beta}_r,\beta_{c_{nom}}\right)$: Reduced modelor biased model

I. Parameter Estimation by taking into account the Bias

- ***Sensitivity expressions:***

  - To "unknown" parameters :

$$X_r = \frac{\partial F(t, \beta_r, \beta_c)}{\partial \beta_r}$$

  - To "known" parameters:

$$X_c = \frac{\partial F(t, \beta_r, \beta_c)}{\partial \beta_c}$$

Relation (1) shows that:

$$\tilde{X}_r = \frac{\partial \tilde{F}(t, \hat{\beta}_r, \beta_{c_{nom}})}{\partial \hat{\beta}_r} = X_r$$

I. Parameter Estimation taking into account the Bias

"Unknown" Parameter Estimation: O.L.S Method

$$S = \sum_{i=1}^{nt} \left(F(t_i, \beta_r, \beta_c) - \tilde{F}\left(t_i, \hat{\beta}_r, \beta_{c_{nom}}\right) \right)^2$$

$$\frac{\partial S}{\partial \hat{\beta}_r} = 0 \Rightarrow \tilde{X}_r . \left(F(t, \beta_r, \beta_c) - \tilde{F}\left(t, \hat{\beta}_r, \beta_{c_{nom}}\right) \right) = 0$$

Matrix formulation: $$\tilde{X}_r^{\,t} . Y = 0$$

With: $$Y = F(t, \beta_r, \beta_c) - \tilde{F}\left(t, \hat{\beta}_r, \beta_{c_{nom}}\right)$$

I. Parameter Estimation by taking into account the Bias

$$\tilde{F}\left(t,\hat{\beta}_r,\beta_{c_{nom}}\right)=F\left(t,\beta_r,\beta_c\right)+\left(X_r \middle| X_c\right)\begin{pmatrix} b_{\beta_r}=\hat{\beta}_r-\beta_r \\ e_{b_c}=\beta_{c_{nom}}-\beta_c \end{pmatrix}$$

$$+ \quad \tilde{X}_r^{\,t}.Y=0$$

$$\Downarrow$$

$$\tilde{X}_r^{\,t}.Y=0=-\tilde{X}_r^{\,t}X_r.b_{\beta_r}-\tilde{X}_r^{\,t}X_c.e_{b_c}$$

$$\Downarrow$$

!!!

$$b_{\beta_r}=-\left(\tilde{X}_r^{\,t}X_r\right)^{-1}.\tilde{X}_r^{\,t}X_c.e_{b_c}$$

Determination of bias is not possible because $X_c.e_{b_c}$ is unknown !!!

I. Parameter Estimation by taking into account the Bias

$X_c . e_{b_c}$ Can be estimated from the Residuals curve:

$$r = \left(X_r . \left(\tilde{X}_r^t X_r \right)^{-1} . \tilde{X}_r^t - Id \right) . X_c . e_{b_c}$$

Assumptions: $\tilde{X}_r = X_r$

???

$$\left(\tilde{X}_r . \left(\tilde{X}_r^t \tilde{X}_r \right)^{-1} . \tilde{X}_r^t - Id \right) . X_c . e_{b_c} = r$$

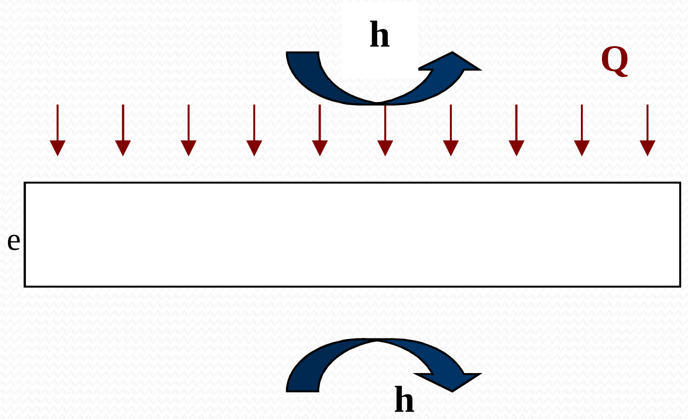
"Known" "Known"

$$AX = B$$

Linear System to Solve …

II. Application to Flash Method

- Principle of the "Flash" method



Heat Eq.:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t}$$

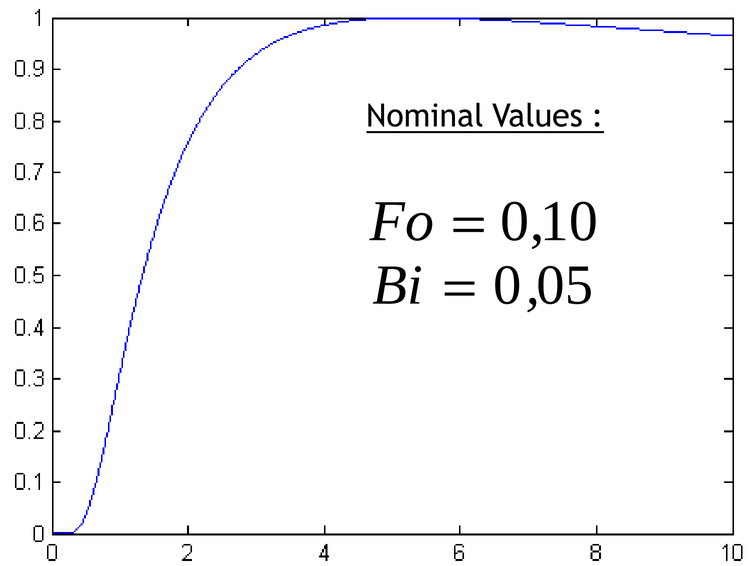
BC et IC:

$$\begin{cases} \text{at } t = 0, & T = 0 \\ \text{in } x = 0, & \lambda \frac{\partial T}{\partial x} = hT_0 - \varphi(t) \\ \text{in } x = e, & -\lambda \frac{\partial T}{\partial x} = hT_e \end{cases}$$

Solution if given by: $T = f\left(\frac{he}{\lambda}, \frac{a}{e^2}, t\right)$

Two parameters:

- Fourier number ("unknown" parameter)
- Biot number ("known" parameter)



II. Application to Flash Method

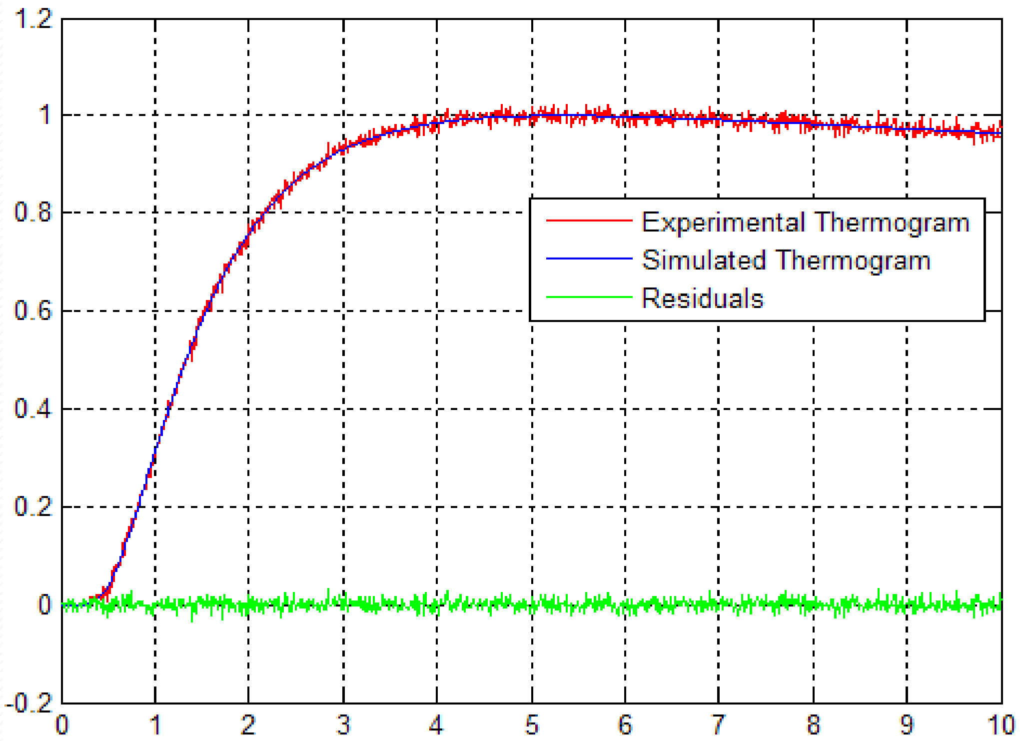
Unbiased Model $\hat{\beta} \approx \beta$ → $\mathbf{T}(\mathbf{t},\hat{\beta}) = \mathbf{T}(\mathbf{t},\beta) + \mathbf{X}|_{\beta}(\hat{\beta} - \beta)$

→ $$\hat{\beta} = \beta + \left(\mathbf{X}^t\mathbf{X}\right)^{-1}\mathbf{X}^t\varepsilon(\mathbf{t})$$

$$\begin{cases} \mathbf{E}(\hat{\beta}) = \beta \\ \mathbf{V}(\hat{\beta}) = \sigma_b^{\,2}\left(\mathbf{X}^t\mathbf{X}\right)^{-1} \end{cases}$$

Residuals $\mathbf{r}(t_i) = \mathbf{Y}(t_i,\beta) - \mathbf{T}(t_i,\hat{\beta})$

$$\begin{cases} \mathbf{E}(\mathbf{r}) = 0 \\ \mathbf{V}(\mathbf{r}) = \sigma_b^{\,2} \end{cases}$$



$$\hat{\beta} = \beta + \left(X^t X\right)^{-1} X^t \varepsilon(t)$$

$$E\left(\hat{\beta}\right) = \begin{pmatrix} 0{,}099 \\ 0{,}049 \end{pmatrix} \qquad \beta = \begin{pmatrix} 0{,}10 \\ 0{,}05 \end{pmatrix} \quad \Longrightarrow \quad E\left(\hat{\beta}\right) = \beta$$

$$V\left(\hat{\beta}\right) = \begin{pmatrix} 0{,}024.10^{-9} \\ 0{,}907.10^{-9} \end{pmatrix} \qquad \sigma_b^{\ 2}\left(X^t X\right)^{-1} = \begin{pmatrix} 0{,}024.10^{-9} \\ 0{,}900.10^{-9} \end{pmatrix} \quad \Longrightarrow \quad V\left(\hat{\beta}\right) = \sigma_b^{\ 2}\left(X^t X\right)^{-1}$$

Residuals

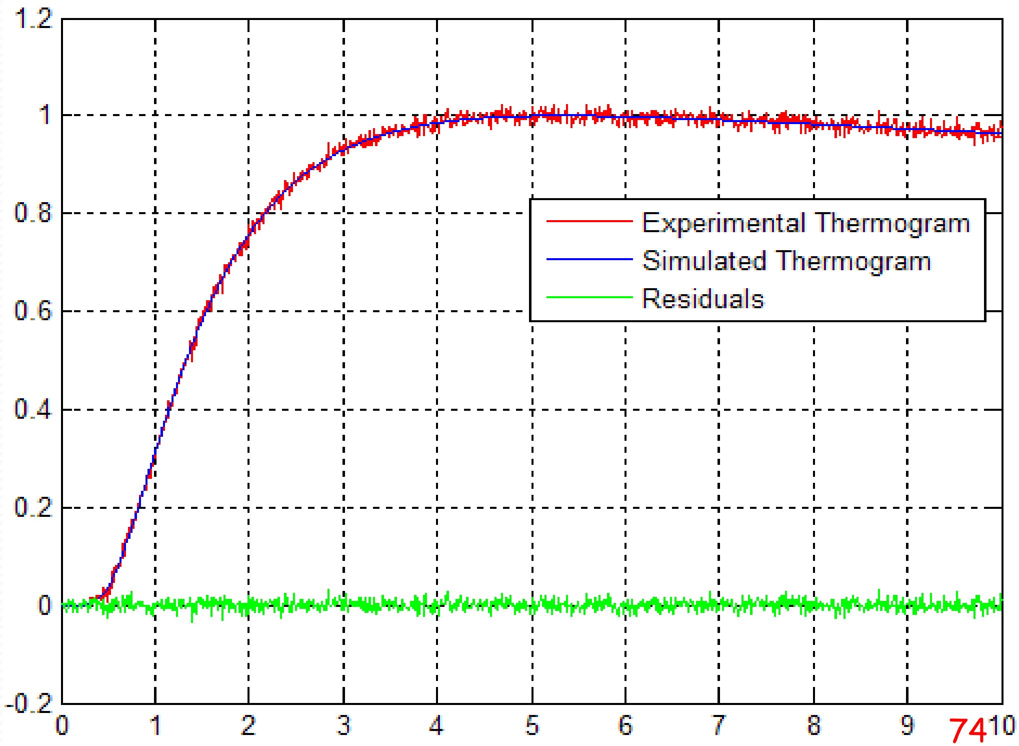
$$r(t_i) = Y(t_i, \beta) - T\left(t_i, \hat{\beta}\right)$$

$$E(r) = 5{,}902.10^{-21}$$

$$\Longrightarrow \quad E(r) = 0$$

$$V(r) = 9{,}989.10^{-5} \qquad \sigma_b^{\ 2} = 1.10^{-4}$$

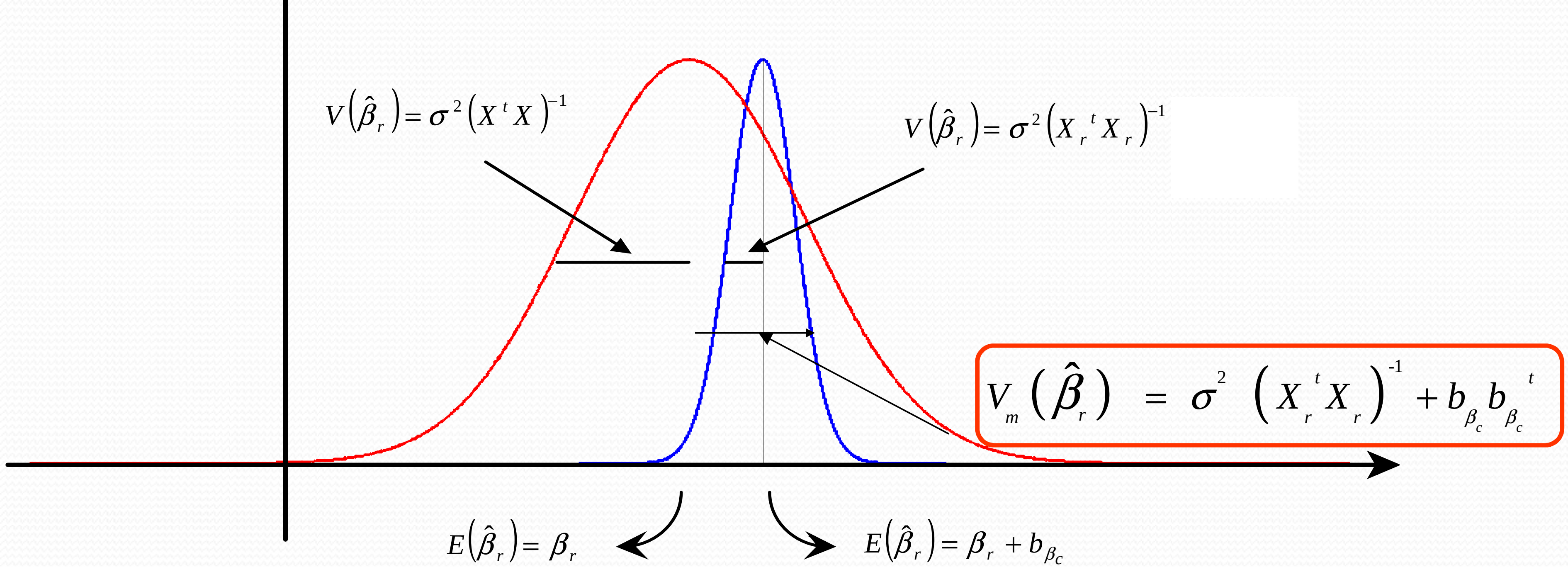
$$\Longrightarrow \quad V(r) = \sigma_b^{\ 2}$$



Biased Model

Parameter Estimation Improvement

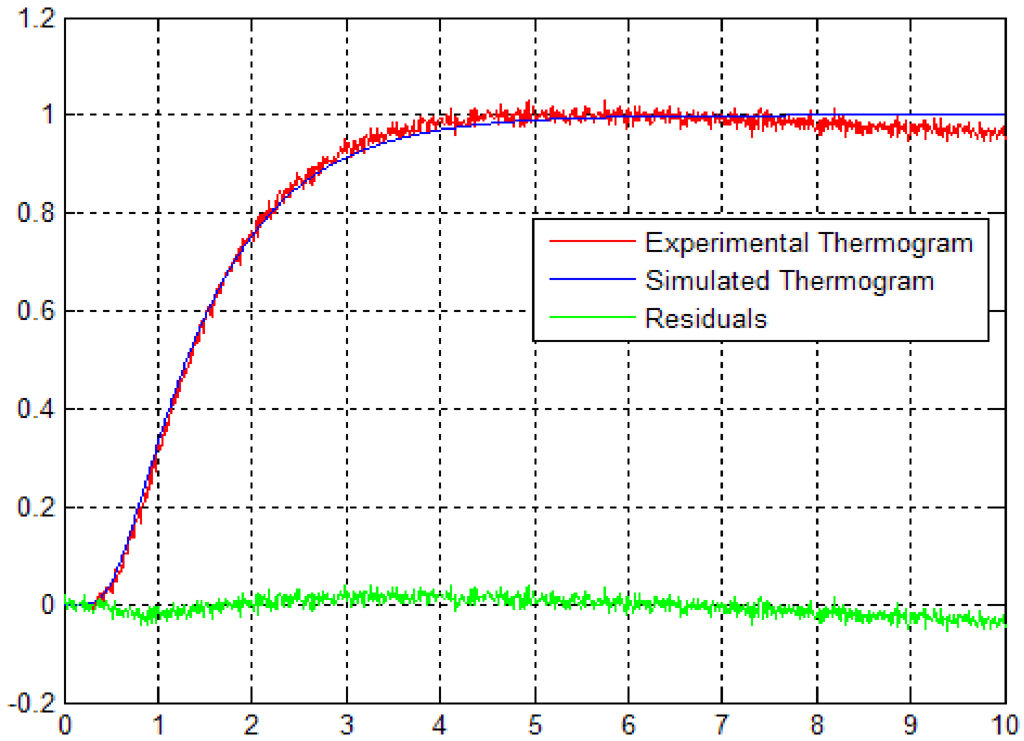
- Measurement noise Reduction
- Number of parameters Reduction
  But Introduction of a **bias**



Variance calculated from true values $\mathbf{V}(\hat{\beta}_r) = \sigma_b^2 (\mathbf{X}_r^t \mathbf{X}_r)^{-1}$

Variance calculated from the biased parameters $\mathbf{V}_m(\hat{\beta}_r) = \sigma_b^2 (\mathbf{X}_r^t \mathbf{X}_r)^{-1} + \mathbf{b}_{\beta_c} \mathbf{b}_{\beta_c}^t$

Biased Model



$$\begin{cases} \boldsymbol{\beta} = (\boldsymbol{\beta}_r | \boldsymbol{\beta}_c) \\ \mathbf{X} = (\mathbf{X}_r | \mathbf{X}_c) \end{cases}$$

$$\mathbf{X}_r{}^t (\mathbf{Y} - \mathbf{T}) = 0$$

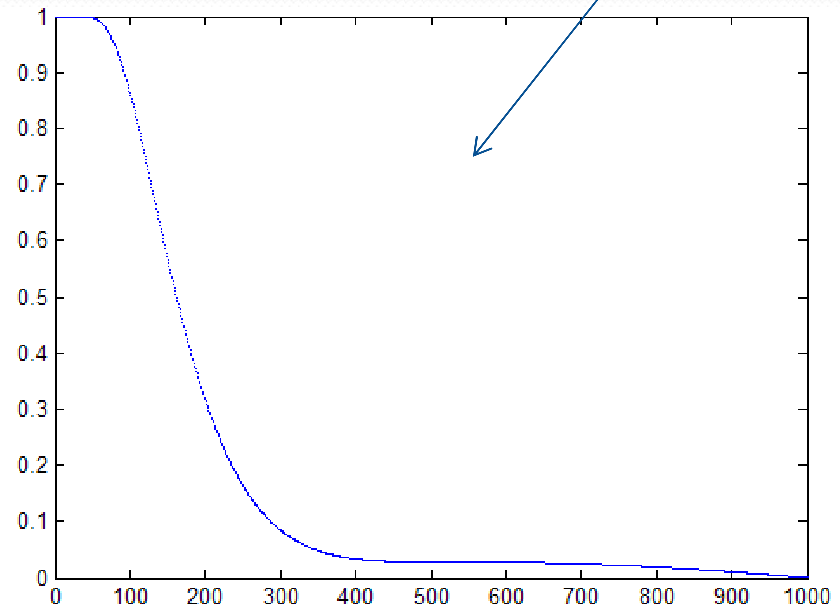
Errror on the assumed “known” parameters: $\mathbf{e}_{\beta_c} = \boldsymbol{\beta}_{c_{nom}} - \boldsymbol{\beta}_c$

$$\mathbf{T}\left(\mathbf{t}, \hat{\boldsymbol{\beta}}_r, \boldsymbol{\beta}_{c_{nom}}\right) = \mathbf{T}(\mathbf{t}, \boldsymbol{\beta}_r, \boldsymbol{\beta}_c) + (\mathbf{X}_r | \mathbf{X}_c) \begin{pmatrix} \hat{\boldsymbol{\beta}}_r - \boldsymbol{\beta}_r \\ \mathbf{e}_{b_c} \end{pmatrix}$$

$$\Rightarrow \quad \hat{\boldsymbol{\beta}}_r = \boldsymbol{\beta}_r + \left(\mathbf{X}_r{}^t \mathbf{X}_r\right)^{-1} \mathbf{X}_r{}^t \boldsymbol{\varepsilon} - \left(\mathbf{X}_r{}^t \mathbf{X}_r\right)^{-1} \mathbf{X}_r{}^t \mathbf{X}_c \mathbf{e}_{\beta_c}$$

II. Application to Flash Method

$$r = \left(\tilde{X}_r . \left(\tilde{X}_r^{\,t} \tilde{X}_r \right)^{-1} . \tilde{X}_r^{\,t} - Id \right) . X_c . e_{b_c}$$

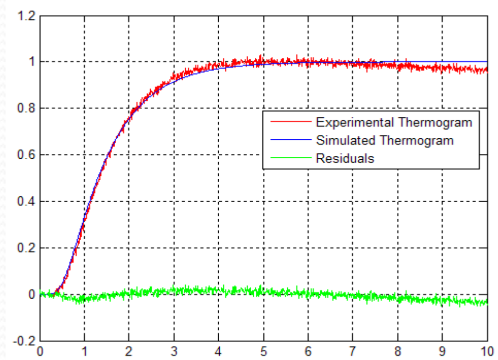


Matrix determinant versus Matrix Rank

Matrix inversion can not be implemented

(Determinant = 0)

III. Parameters Estimation using a Time Variable Interval



We have previously shown:

$$b_{\beta_r} = -\left(\tilde{X}_r^{\,t} X_r\right)^{-1} . \tilde{X}_r^{\,t} X_c . e_{b_c}$$

- If $e_{b_c} = 0$, then $b_{\beta_r} = 0$ and residuals curve is unsigned ($r = 0$).

- If $e_{b_c} \neq 0$, then residuals curve is signed ($r \neq 0$)
  - b_{β_r} is null if $\tilde{X}_r^{\,t} . X_c = 0$ (uncorrelated parameters).

    In this case, $r = -X_c . e_{b_c}$
  - b_{β_r} is different to zero if $\tilde{X}_r^{\,t} . X_c \neq 0$

    In this case, $r = -X_c . e_{b_c} - X_r . b_{\beta_r}$

III. Parameters Estimation using a Time Variable Interval

Time intervals truncated to time t1 and t2 will be denoted : $[0-t_1]$ $[0-t_2]$

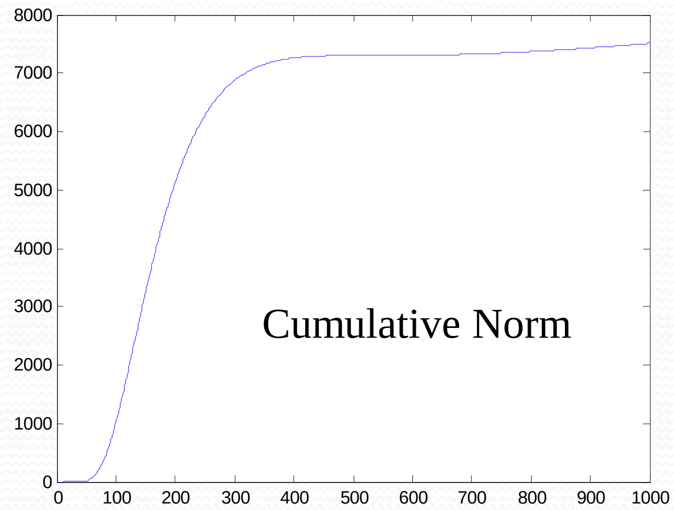
Approximation:

$$\tilde{X}_{r2}{}^t X_{r2} = \tilde{X}_{r1}{}^t X_{r1} + \underbrace{\int_{t=t_1}^{t_2} \tilde{X}_{r1}{}^t X_{r1}\, dt}_{\to 0} \simeq \underset{t_2 \to t_1 \,/\, t_1 \to t_2}{\tilde{X}_{r1}{}^t X_{r1}}$$

Bias:

$$b_{\beta_{r1}} = \hat{\beta}_{r_1} - \beta_r = -\left(\tilde{X}_{r1}{}^t X_{r1}\right)^{-1} . \tilde{X}_{r1}{}^t X_{c1} . e_{b_c}$$

$$b_{\beta_{r2}} = \hat{\beta}_{r_2} - \beta_r = -\left(\tilde{X}_{r2}{}^t X_{r2}\right)^{-1} . \tilde{X}_{r2}{}^t X_{c2} . e_{b_c}$$



Bias variation:

$$\Delta b_{\beta_{r\,2-1}} = \hat{\beta}_{r_2} - \hat{\beta}_{r_1} = -\left(\tilde{X}_{r2}{}^t X_{r2}\right)^{-1} . \tilde{X}_{r2}{}^t X_{c2} . e_{b_c} + \left(\tilde{X}_{r1}{}^t X_{r1}\right)^{-1} . \tilde{X}_{r1}{}^t X_{c1} . e_{b_c}$$

$$\boxed{\Delta b_{\beta_{r\,2-1}} = \hat{\beta}_{r_2} - \hat{\beta}_{r_1} = -\left(\tilde{X}_{r1}{}^t X_{r1}\right)^{-1} . \left[\tilde{X}_r{}^t (t_2) X_c (t_2) . e_{b_c}\right]}$$

III. Parameters Estimation using a Time Variable Interval

Setting: $t_m = (t_1 + t_2)/2$

Bias difference can be written:

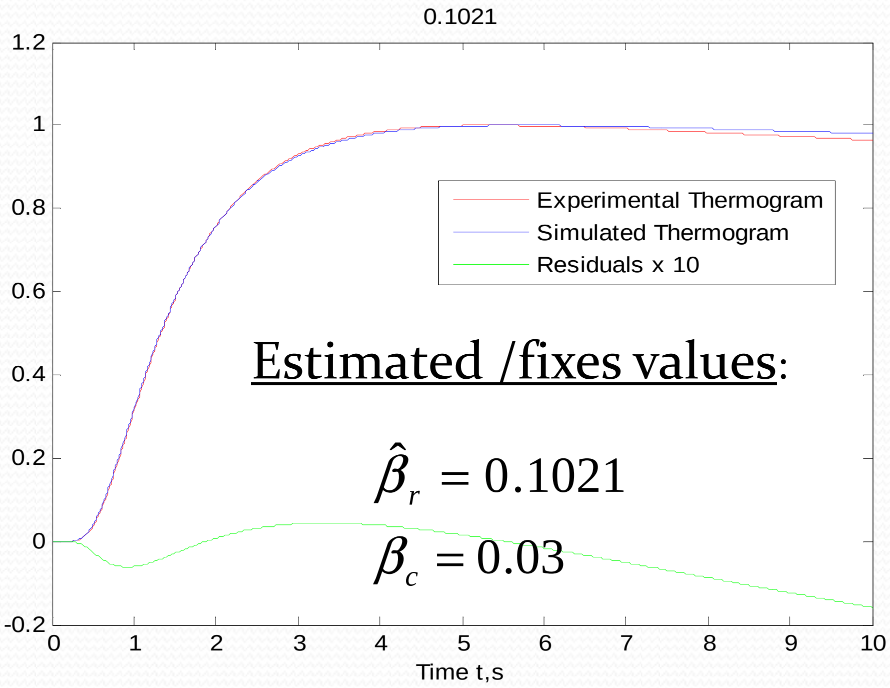
$$\Delta b_{\beta_{r\,2-1}} = \hat{\beta}_{r_2} - \hat{\beta}_{r_1} = -\left(\tilde{X}_{r1}^{\,t} X_{r1}\right)^{-1}.\left[\tilde{X}_{r}^{\,t}(t_m) X_c(t_m).e_{b_c}(n_2 - n_1)\right]$$

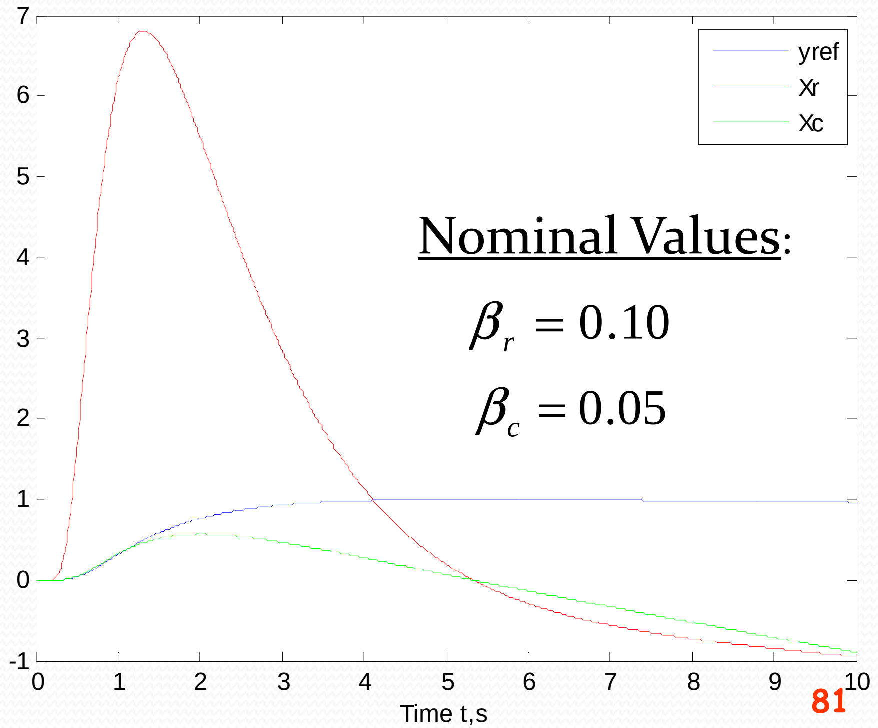
$$X_c(t_m).e_{b_c} = -\frac{\left(\hat{\beta}_{r_2} - \hat{\beta}_{r_1}\right)\left(\tilde{X}_{r1}^{\,t} X_{r1}\right)}{\tilde{X}_{r}^{\,t}(t_m)(n_2 - n_1)} \Rightarrow X_c e_{\beta_c} = -E(r) - X_r b_{\beta_r}$$

$$b_{\beta_r} = \frac{\left(-Y(t_m) - X_c(t_m).e_{b_c}\right)}{X_r(t_m)}$$

III. Parameters Estimation using a Time Variable Interval

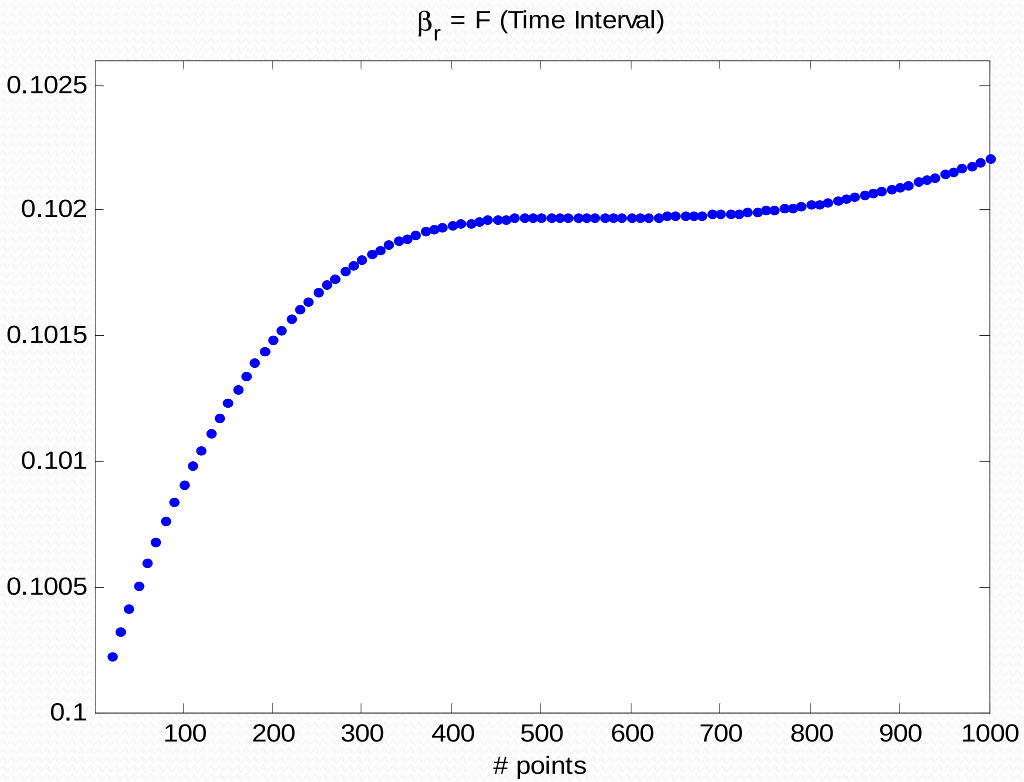
- **Bias Estimation using a time varying estimation interval in the case of the Flash Method**





III. Parameters Estimation using a Time Variable Interval

- **Bias Estimation using a time varying estimation interval in the case of the Flash Method**



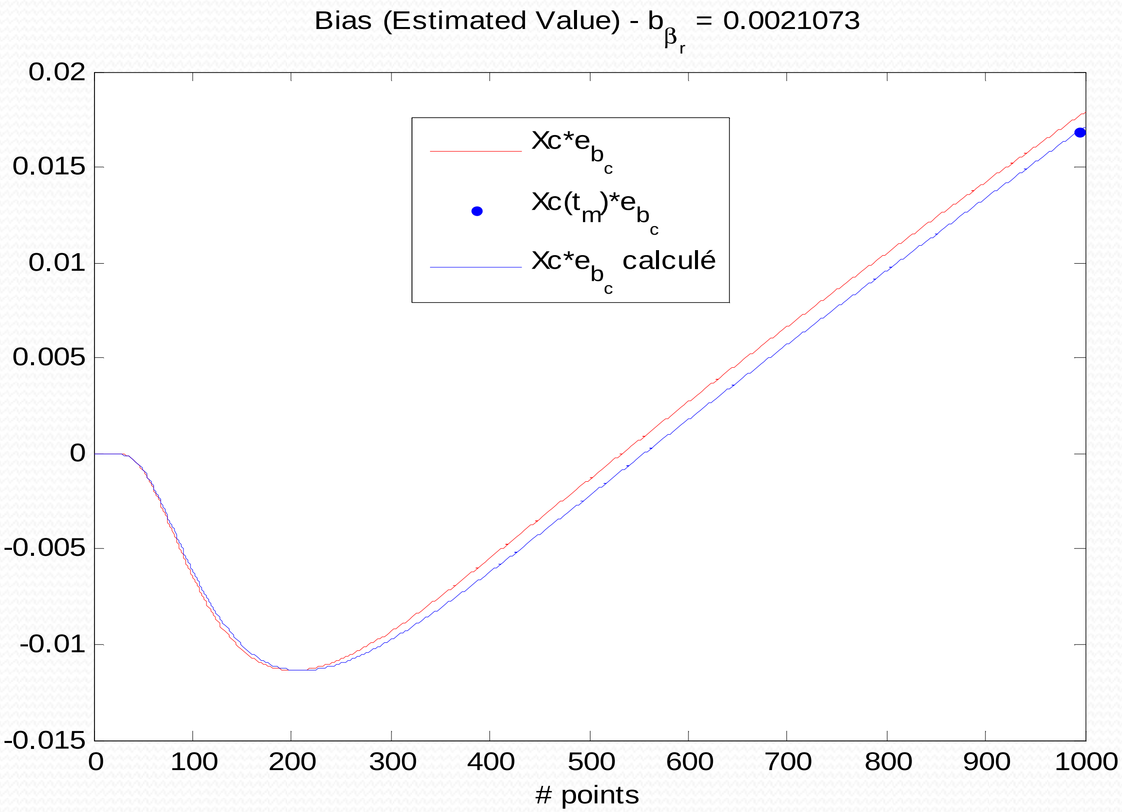
$\hat{\beta}_r$

is a function of the time interval length

It exhibits the presence of a bias b_{β_r}

III. Parameters Estimation using a Time Variable Interval

- **Bias Estimation using a time varying estimation interval in the case of the Flash Method**



Estimated Bias = 0.0021073

Expected Bias = 0.0021

- Efficient technique
- How to chose the optimal length of the time interval ?

GENERAL CONCLUSIONS

- **No General Rules** in the Case of a Non-Linear Problems but a Methodology and different Tools exist

- **Different Aspects** specific to Non-Linear Problems must be taken into account to improve the Parameters Estimation (**both** in **Experimental** and **Theoretical** Points of View)

- The **Using of a Reduced Model** is a Solution but **a particular attention must be paid to the Bias** on Estimated Parameters

- **Bias can be estimated from "Known" Quantities** (Sensitivity to Estimated Parameters and Residuals Curve)

Thank You For Your Attention !!!