Lecture 1: Getting started with problematic inversions

Denis Maillet, Yvon Jarny, Daniel Petit, Olivier Fudym, Philippe Le Masson

LEMTA Nancy - LTN Nantes - Institut P' Poitiers - Ecole Mines Albi - LIMATB Lorient

Example 1: Square system of linear equations

Example 2: Different inverse problems fo steady state 1D heat transfer through a wall

Example 1: Square system of linear equations

$$10\ x_1 - 21\ x_2 = 9$$
$$39\ x_1 - 81\ x_2 = 1$$

scalar relationship $\rightarrow$ vector

$$\boldsymbol{S} = \begin{bmatrix} 10 & -21 \\ 39 & -81 \end{bmatrix} \qquad \boldsymbol{x} = \boldsymbol{x}^{exact} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow \boldsymbol{y}_{mo} = \boldsymbol{S}\ \boldsymbol{x}^{exact} = \begin{bmatrix} 9 \\ 36 \end{bmatrix}$$

Direct problem: input (known) output (calculated)

Model : $\boldsymbol{y}_{mo} = \eta\ (\boldsymbol{x})$

Structure of model

$$\mathbf{S} = \begin{bmatrix} 10 & -21 \\ 39 & -81 \end{bmatrix} \qquad \mathbf{y}_{mo} = \mathbf{S}\,\mathbf{x}^{exact} = \begin{bmatrix} 9 \\ 36 \end{bmatrix}$$

inverse problem: $\mathbf{y}_{mo}$: data (known); $\mathbf{x}^{exact}$: unknown

Solution with exact data $\mathbf{y}_{mo}$:

$$\mathbf{x} = \mathbf{S}^{-1}\mathbf{y}_{mo} = \mathbf{x}^{exact} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

No problem !

Solution with noisy data $\mathbf{y}$:

$$\mathbf{y} = \mathbf{y}_{mo} + \boldsymbol{\varepsilon} = \begin{bmatrix} 9.1 \\ 35.7 \end{bmatrix} \qquad \boldsymbol{\varepsilon} = \begin{bmatrix} 0.1 \\ -0.3 \end{bmatrix}$$

$\boldsymbol{\varepsilon}$: noise

$\approx 1\ \%$ of $y_{mo\,1}$

$\approx 1\ \%$ of $y_{mo\,2}$

Noise amplification !

$$\begin{bmatrix} 10 & -21 \\ 39 & -81 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 9{,}1 \\ 35.7 \end{bmatrix}$$

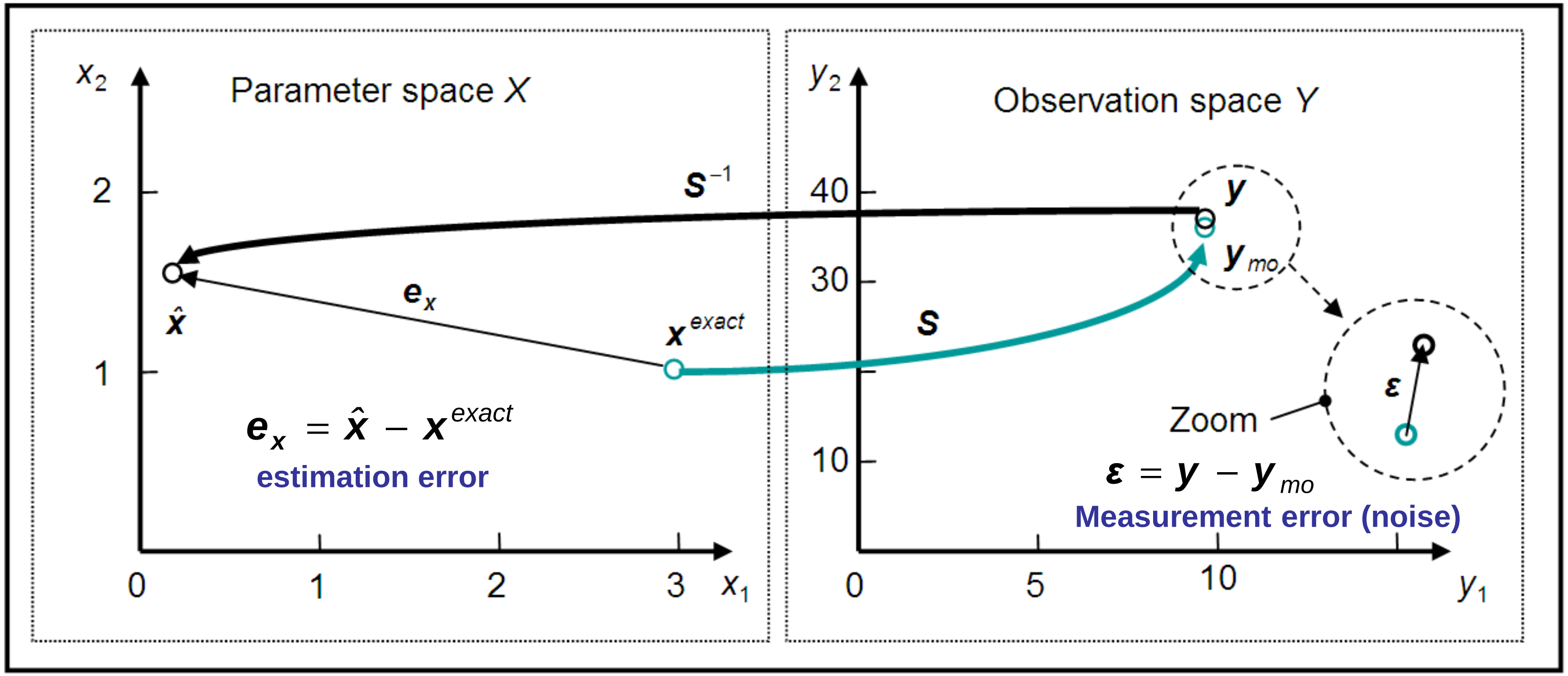
$$\hat{\mathbf{x}} = \mathbf{x}^{exact} + \mathbf{e}_x = \begin{bmatrix} 1.40 \\ 0.233 \end{bmatrix}$$

$\hat{\mathbf{x}}$: estimate; $\mathbf{e}_x$: estimation error

53 % error for x_1

77 % error for x_2

Euclidian distance (L2 norm): $\|\boldsymbol{u}\| = \left(\sum_{i=1}^{2} u_j^2\right)^{1/2}$



$$k_a(\boldsymbol{\varepsilon}) = \frac{\|\boldsymbol{S}^{-1}\boldsymbol{\varepsilon}\|}{\|\boldsymbol{\varepsilon}\|} = \frac{\|\boldsymbol{e}_x\|}{\|\boldsymbol{\varepsilon}\|} = \frac{1.774}{0.316} = 5.61$$

$\det(\boldsymbol{S}) = 9$

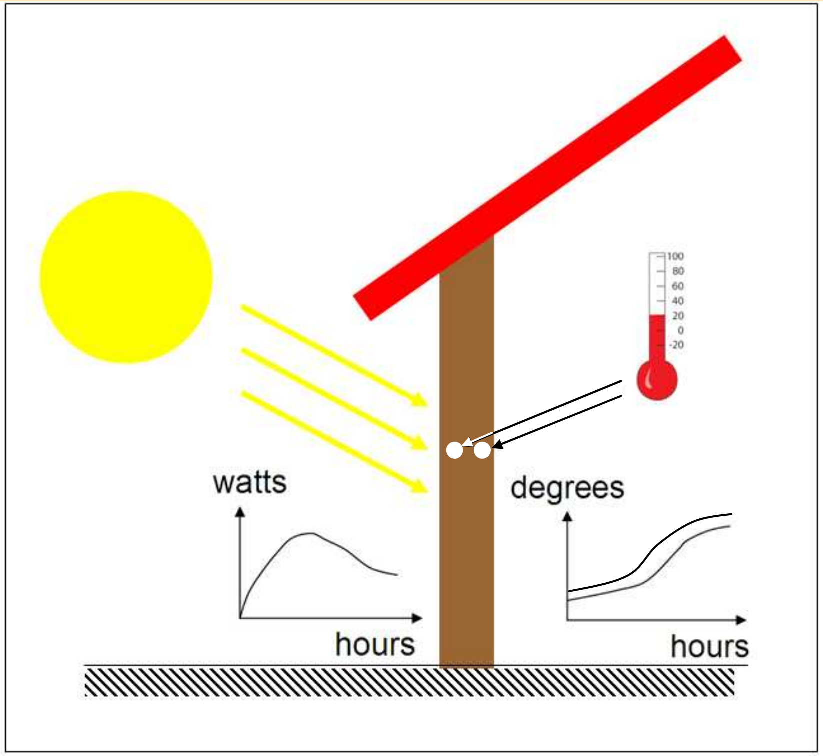
coefficients of amplification of measurement error (leverage)

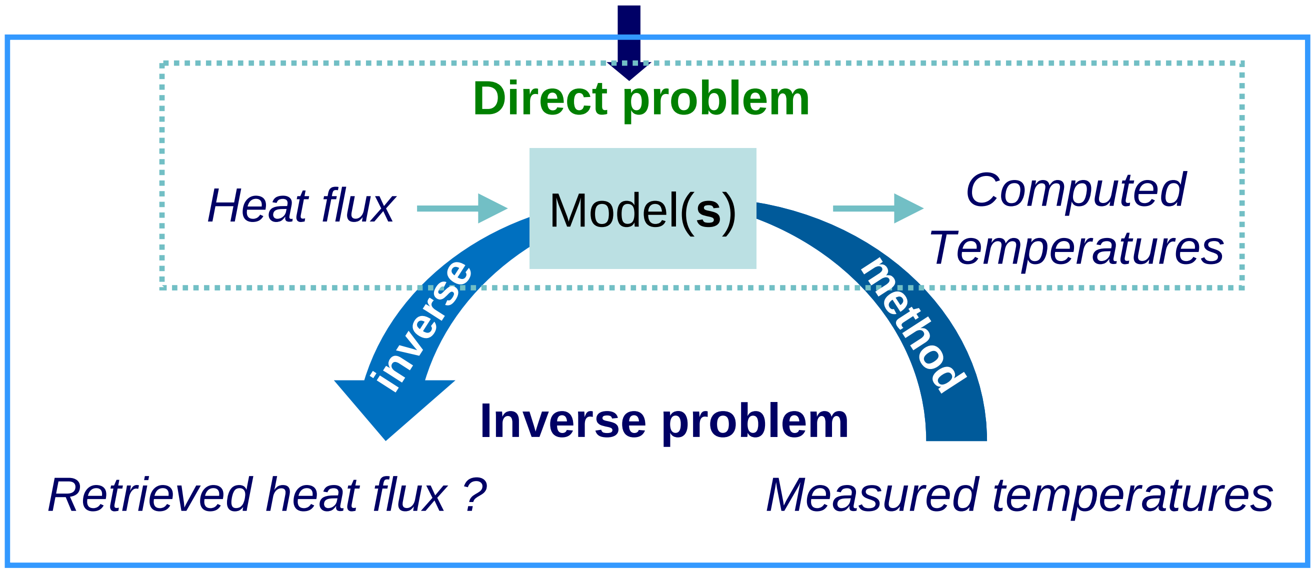
absolute → k_a; relative → k_r

maximum: $k_r(\boldsymbol{\varepsilon}) \le \text{cond}(\boldsymbol{S}) = 958$

$$k_r(\boldsymbol{\varepsilon}) = \frac{\|\boldsymbol{S}^{-1}\boldsymbol{\varepsilon}\| / \|\boldsymbol{S}^{-1}\boldsymbol{y}_{mo}\|}{\|\boldsymbol{\varepsilon}\| / \|\boldsymbol{y}_{mo}\|} = \frac{\|\boldsymbol{e}_x\| / \|\boldsymbol{x}^{exact}\|}{\|\boldsymbol{\varepsilon}\| / \|\boldsymbol{y}_{mo}\|} = \frac{1.774/3.16}{0.316/37.11} = 65.8$$

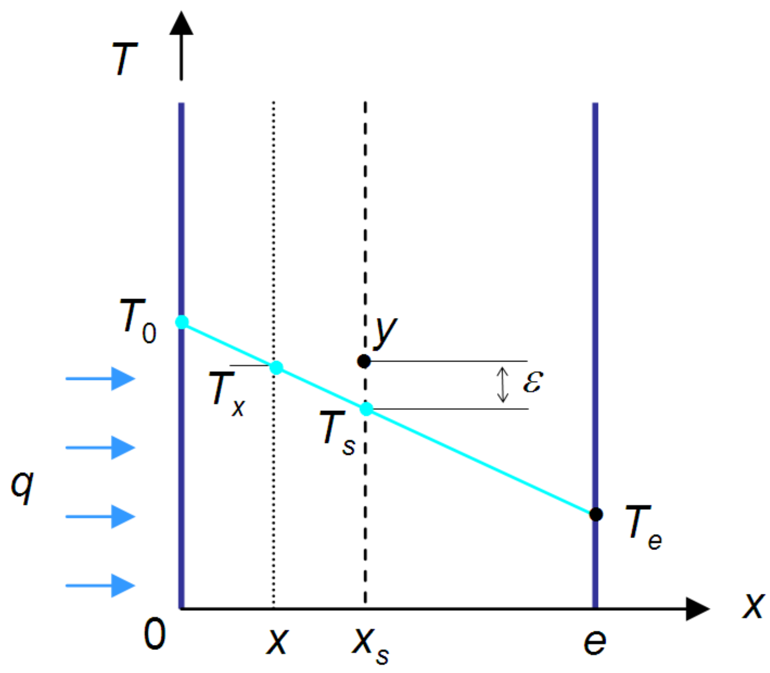
Example of
inverse
heat conduction
problem (IHCP)





Example 2: Different inverse problems fo **steady state 1D** heat transfer through a wall

Physical system with sensors
- *plane wall*
- *2 temperature sensors:*
  1 on rear face (exact measurement T_e)
  *1 inside ($x = x_s$; **noisy** measurement y)*



Thermal model
for exact output of sensor T_s

- *homogeneous material (conductivity λ)*
- *steady state*
- *stimulation q ($x = 0$)*
- *1D heat transfer*
- *no internal source*
- *Fourier law*

Possible objectives
(types of inverse problems)

- *flux q entering the wall ($x = 0$) ?*
- *front face temperature T_0 ?*
- *internal temperature distribution $T(x)$?*
- *conductivity λ ?*

State equations: T ?

$$\frac{\partial^2 T}{\partial x^2} = 0 \quad \text{with} \quad -\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} = q \quad \text{and} \quad T\big|_{x=e} = T_e$$

model structure

$$y_{mo} = T_x = \eta_1\,(x\,;\,q,\,T_0\,,\lambda) \equiv T_0 - q\;x\,/\,\lambda$$

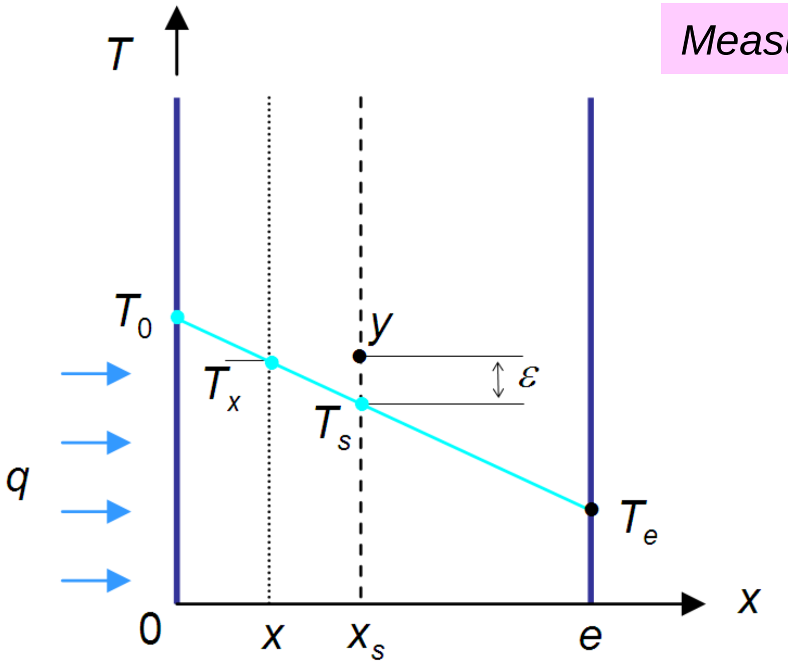
parameters

dependent variable or explained variable

Assumption: λ known, no error for T_e
no error for e and x_s
Objective: find q, T_0, T_x

Output equation: $T_s = \eta_1\,(x_s\,;\,q,\,T_0\,,\lambda)$

$$\equiv \eta_2\left(\frac{x_s}{e},\,T_0,\,T_e\right) = \left(1 - \frac{x_s}{e}\right)T_0 + \frac{x_s}{e}\,T_e$$

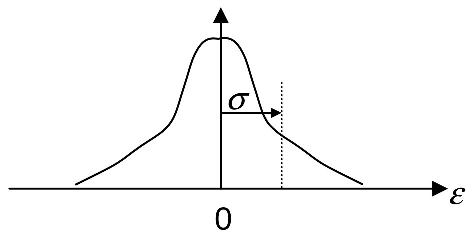


Measurement:

$$y = T_s + \varepsilon \longrightarrow \textit{Random variable}$$

measurement — exact unknown temperature — noise

p.d.f: $E(\varepsilon) = 0 \quad E(\varepsilon^2) = \sigma^2$



Estimation = exact matching:

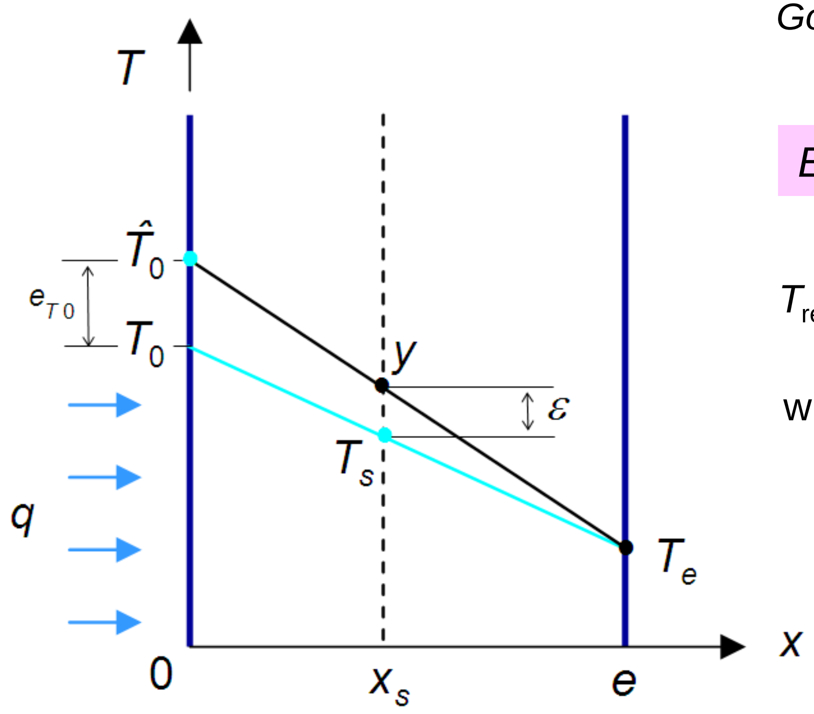
estimate

$$y = T_s = \eta_2\left(\frac{x_s}{e},\,\hat{T}_0,\,T_e\right)$$

Solution of inverse problem: estimation of T_0

$$\hat{T}_0 = \frac{1}{1-x_s^*}\, y - \frac{x_s^*}{1-x_s^*}\, T_e \qquad \text{with} \qquad x_s^* = x_s / e$$

Estimation error: $e_{T0} = \varepsilon / (1 - x_s^*) \quad \Rightarrow \quad E(e_{T0}) = 0 \quad \text{and} \quad \sigma_0 = \sigma / (1 - x_s^*)$



Good estimation of T_0 for shallow measurement

Estimation of T (x) :

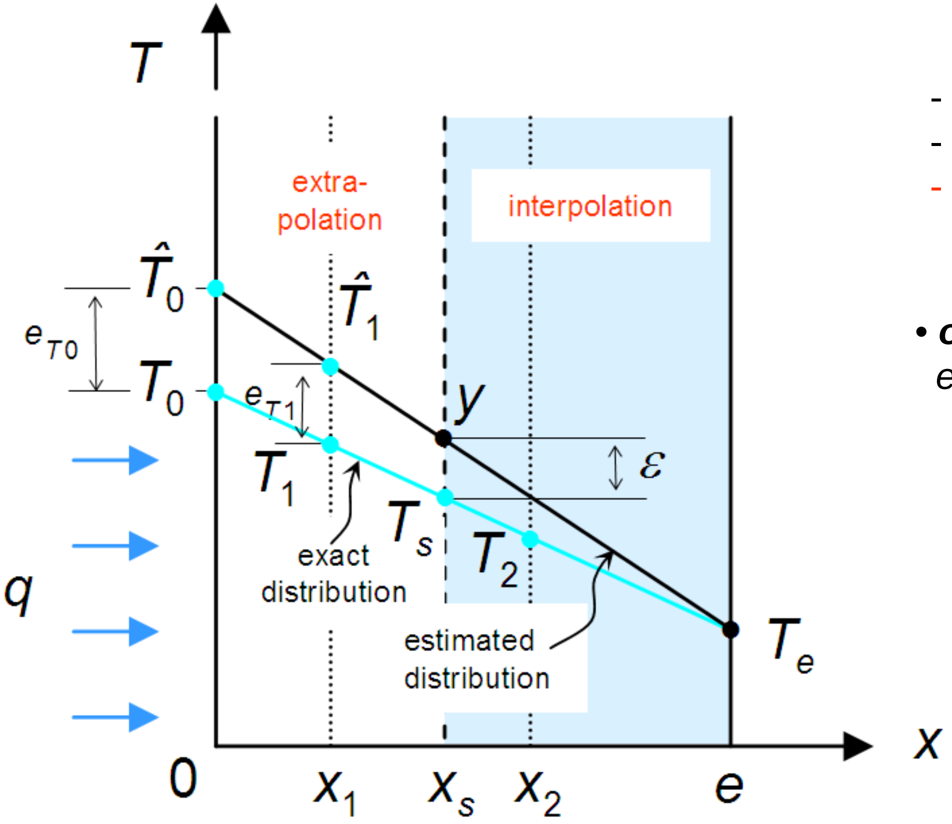
$$T_{recalc}(x) = \eta_2(x^*, \hat{T}_0) = \hat{T}_x = \frac{1-x^*}{1-x_s^*}\, y + \frac{x^* - x_s^*}{1-x_s^*}\, T_e$$

with $x^* = x / e$

Estimation error:

$$e_{Tx} = K\,\varepsilon \quad \Rightarrow \quad \sigma_{Tx} = K\,\sigma \quad \text{with} \quad K = \frac{1-x^*}{1-x_s^*}$$

$$\sigma_{Tx} = K\ \sigma \qquad \text{with} \qquad K = \frac{1 - x^*}{1 - x_s^*}$$



Two regions for estimation of T_x

- ***in between*** *measurements points* $x \in [\ x_s, e]$
  interpolation = attenation of error

$$K \le 1$$

well-posed problem (Hadamard, 1902):

- *solution exists*
- *it is unique*
- *it depends continuously of the data*

- ***outside*** *measurements points interval* $x \in [0\ ,\ x_s[$
  extrapolation = attenation of error

$$K > 1$$

ill-posed problem (Hadamard, 1902)

$$x_s \to e \Rightarrow K \to \infty \quad \forall x \neq e$$

Very bad design !

Estimation of flux q :

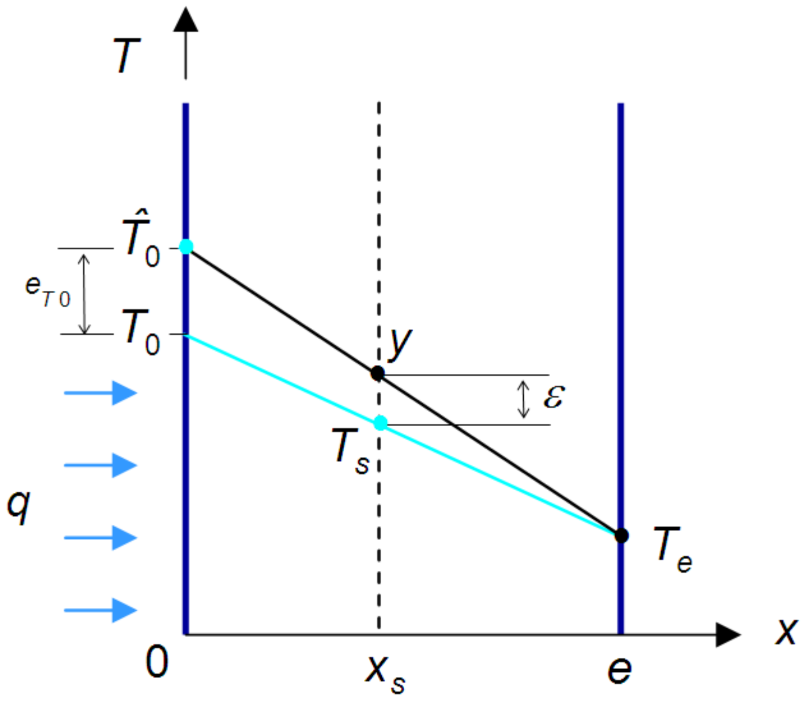
$$\hat{q} = \lambda \frac{y - T_e}{e - x_s}$$

Estimation error:

$$e_q = \frac{\lambda}{e - x_s}\,\varepsilon \Rightarrow \sigma_q = \frac{\lambda}{e - x_s}\,\sigma \Rightarrow \sigma_q / q = \frac{1}{1 - x_s^*}\,\frac{1}{SNR}$$

$$SNR = (T_e - T_0)/\sigma$$

signal over noise ratio



Numerical application:

$e = 0.2$ m - $\lambda = 1$ W.m^{-1}.K^{-1} - $T_0 - T_e = 30$°C
$x_s = 0.18$ m - $\sigma = 0.3$°C

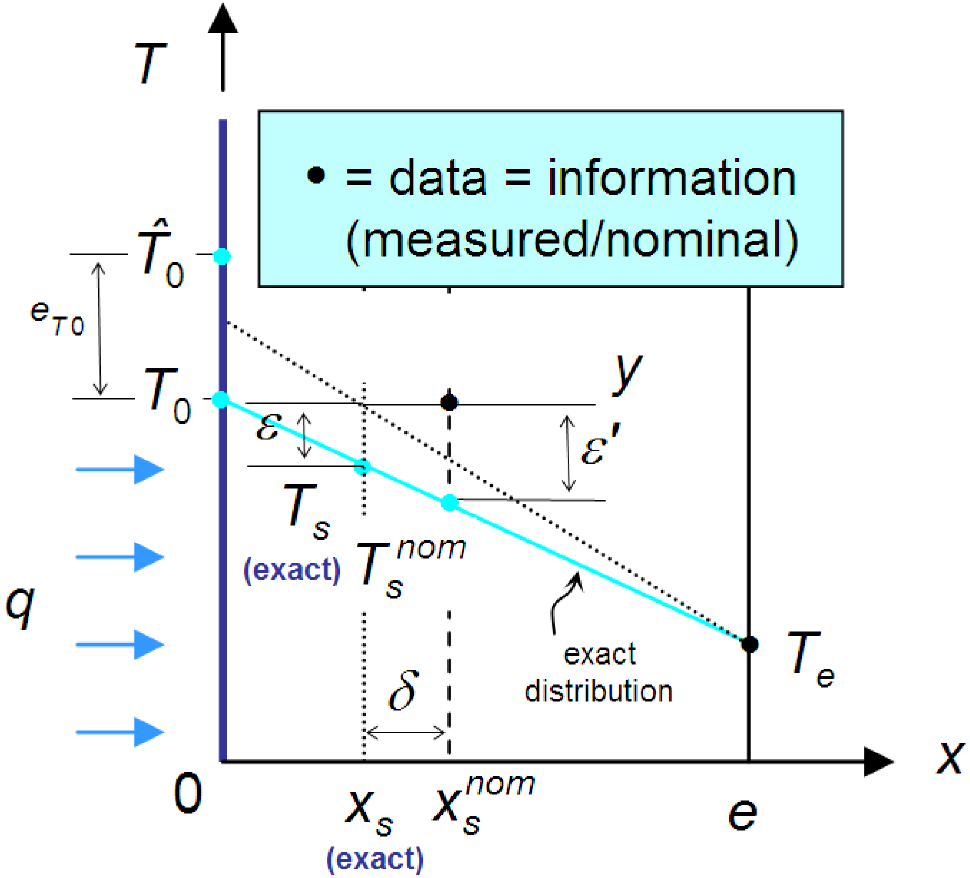
$$\Rightarrow SNR = 100 \text{ and } \sigma_q / q = 10\ \%$$

mid-slab measurement:

$$x_s = e/2 = 0.10\,\text{m} \Rightarrow \sigma_q / q = 2\ \%$$

Errors for parameters "assumed to be known"

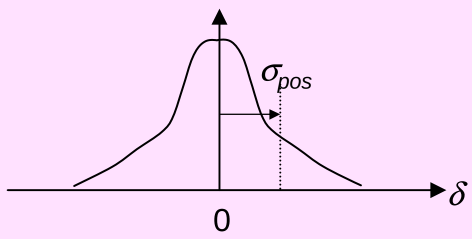
Assumption: λ known, no error for T_e
no error for e
error for x_s
Objective: find q, T_0, T_x



$$x_s^{nom} = x_s + \delta$$

- x_s^{nom}: **nominal (« *a priori* ») location of sensor (deterministic)**
- x_s: **exact location (random)**
- δ: **location error (random)**

p.d.f: $E(\delta) = 0$ $\quad E(\delta^2) = \sigma_{pos}^2$



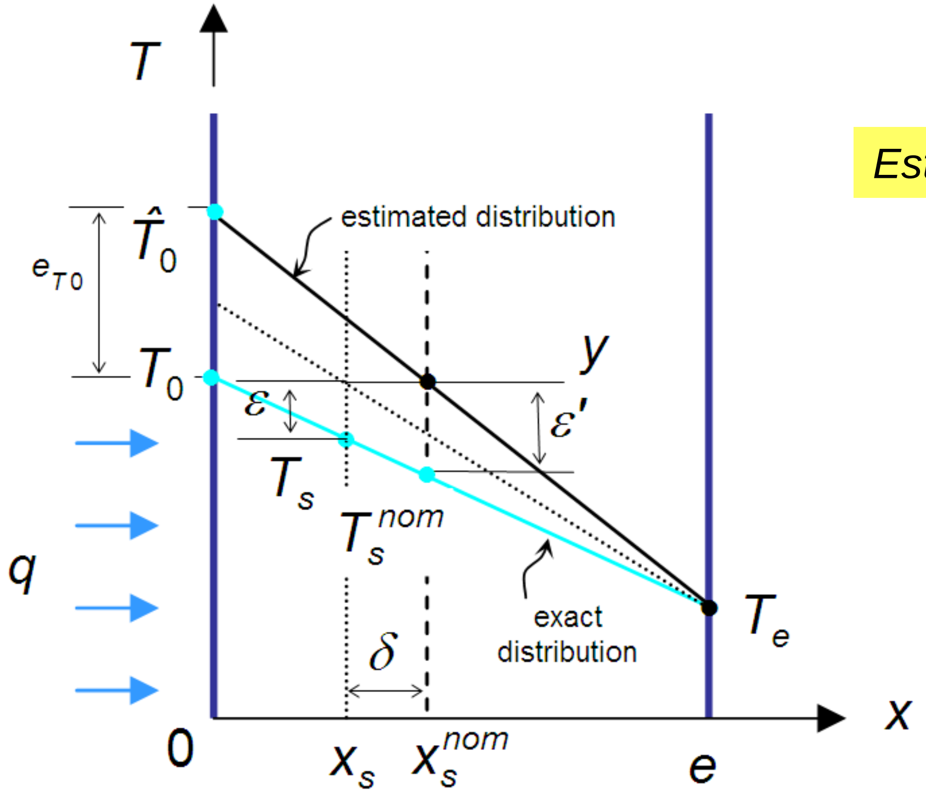
Signal and model : $y = \eta_2\ (x_s / e, T_0, T_e) + \varepsilon = \eta_2\ (x_s^{nom} / e, T_0, T_e) + \varepsilon'$

with $\varepsilon' = \delta\ (T_0 - T_e)/e + \varepsilon$

Equivalent temperature noise :

$$\sigma'^2 = \text{var}\ (\varepsilon') = \sigma^2 + ((T_0 - T_e)/e)^2\ \sigma_{pos}^2$$
$$= \sigma^2\ (1 + SNR^2 / R_{pos}^2)$$

with $R_{pos} = e / \sigma_{pos}$



Estimation error:

$$\sigma_q / q = \frac{1}{1 - x_s^*}\ \frac{1}{SNR'}$$

$$SNR' = (T_e - T_0)/\sigma'$$

Numerical application:

$e = 0.2$ m - $\lambda = 1$ W.m^{-1}.K^{-1} - $T_e - T_0 = 30$℃
$x_s = 0.18$ m - $\sigma = 0.3$℃ - $\sigma_{pos} = 2$ mm

$\Rightarrow SNR = 100$

$R_{pos} = e / \sigma_{pos} = 200 / 2 = 100$

$\Rightarrow \sigma_q / q = 14.1\ \%$

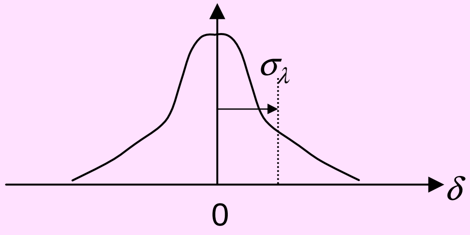
Assumption: no error for T_e
no error for e
no error for x_s
error for λ

Objective: find q, T_0, T_x

$$\lambda^{nom} = \lambda^{exact} + e_\lambda$$

- λ^{nom}: **nominal (« *a priori* ») conductivity (deterministic)**
- λ^{exact}: **exact conductivity (random)**
- e_λ: **conductivity error (random)**

p.d.f: $E(e_\lambda) = 0 \quad E(e_\lambda^2) = \sigma_\lambda^2$



Estimation of flux q :

$$\hat{q} = \lambda^{nom} \frac{y - T_e}{e - x_s} = \frac{\lambda^{exact} + e_\lambda}{e - x_s}(T_s - T_e + \varepsilon)$$

$$= \frac{\lambda^{exact}(T_s - T_e)}{e - x_s}\left(1 + \frac{e_\lambda}{\lambda^{exact}}\right)\left(1 + \frac{\varepsilon}{T_s - T_e}\right)$$

Estimation error:

assumptions: - small $e_\lambda / \lambda^{exact}$
- large *SNR*

$$q^{exact} + e_q = q^{exact}\left(1 + \frac{e_\lambda}{\lambda^{exact}} + \frac{\varepsilon}{T_s - T_e}\right)$$

$$\Rightarrow \quad \frac{e_q}{q^{exact}} = \frac{e_\lambda}{\lambda^{exact}} + \frac{1}{SNR}\frac{\varepsilon}{\sigma}$$

$$\frac{\sigma_q}{q^{exact}} \approx \left(\frac{\sigma_\lambda^2}{(\lambda^{exact})^2} + \frac{1}{SNR^2}\right)^{1/2}$$

Numerical application:

$e = 0.2$ m - $\lambda = 1$ W.m^{-1}.K^{-1} - $T_e - T_0 = 30$°C
$x_s = 0.18$ m - $\sigma = 0.3$°C - $\sigma_{pos} = 0$ mm
$\sigma_\lambda = 0.1$ W.m^{-1}.K^{-1}

$$\Rightarrow \sigma_q / q = 10.1\,\%$$

Thank you for your attention !