KALMAN AND PARTICLE FILTERS

Tutorial 10

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SUMMARY

- State estimation problems
- Kalman filter
- Particle filter
- Examples: Lumped system and 1D Heat conduction
- Applications
- Conclusions

STATE ESTIMATION PROBLEM

State Evolution Model: $\mathbf{x}_k = \mathbf{f}_k(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{v}_{k-1})$

Observation Model: $\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k, \mathbf{n}_k)$

$\mathbf{x} \in R^{n_x}$ = state variables to be estimated

$\mathbf{u} \in R^{n_p}$ = input variable

$\mathbf{v} \in R^{n_v}$ = state noise

$\mathbf{z} \in R^{n_z}$ = measurements

$\mathbf{n} \in R^{n_n}$ = measurement noise

Subscript $k = 1, 2, \ldots,$ denotes an instant t_k in a dynamic problem

STATE ESTIMATION PROBLEM

Definition: The state estimation problem aims at obtaining information about $\mathbf{x}_k$ based on the state evolution model and on the measurements given by the observation model.

BAYESIAN FRAMEWORK

The solution of the inverse problem within the Bayesian framework is recast in the form of statistical inference from the *posterior probability density*, which is the model for the conditional probability distribution of the unknown parameters given the measurements. The measurement model incorporating the related uncertainties is called the *likelihood*, that is, the conditional probability of the measurements given the unknown parameters. The model for the unknowns that reflects all the uncertainty of the parameters without the information conveyed by the measurements, is called the *prior* model.

BAYESIAN FRAMEWORK

The formal mechanism to combine the new information (measurements) with the previously available information (prior) is known as the **Bayes' theorem:**

$$\pi_{posterior}(\mathbf{x}) = \pi(\mathbf{x}|\mathbf{z}) = \frac{\pi(\mathbf{x})\pi(\mathbf{z}|\mathbf{x})}{\pi(\mathbf{z})}$$

where $\pi_{posterior}(\mathbf{x})$ is the posterior probability density, $\pi(\mathbf{x})$ is the prior density, $\pi(\mathbf{z}|\mathbf{x})$ is the likelihood function and $\pi(\mathbf{z})$ is the marginal probability density of the measurements, which plays the role of a normalizing constant.

STATE ESTIMATION PROBLEM

State Evolution Model: $\mathbf{x}_k = \mathbf{f}_k(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{v}_{k-1})$

Observation Model: $\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k, \mathbf{n}_k)$

The *evolution-observation model* is based on the following assumptions :

(i) The sequence $\mathbf{x}_k$ for $k = 1, 2, \ldots$, is a Markovian process, that is,

$$\pi(\mathbf{x}_k \,|\, \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{k-1}) = \pi(\mathbf{x}_k \,|\, \mathbf{x}_{k-1})$$

(ii) The sequence $\mathbf{z}_k$ for $k = 1, 2, \ldots$, is a Markovian process with respect to the history of $\mathbf{x}_k$, that is,

$$\pi(\mathbf{z}_k \,|\, \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k) = \pi(\mathbf{z}_k \,|\, \mathbf{x}_k)$$

(iii) The sequence $\mathbf{x}_k$ depends on the past observations only through its own history, that is,

$$\pi(\mathbf{x}_k \,|\, \mathbf{x}_{k-1}, \mathbf{z}_{1:k-1}) = \pi(\mathbf{x}_k \,|\, \mathbf{x}_{k-1})$$

STATE ESTIMATION PROBLEM

State Evolution Model: $\mathbf{x}_k = \mathbf{f}_k(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{v}_{k-1})$

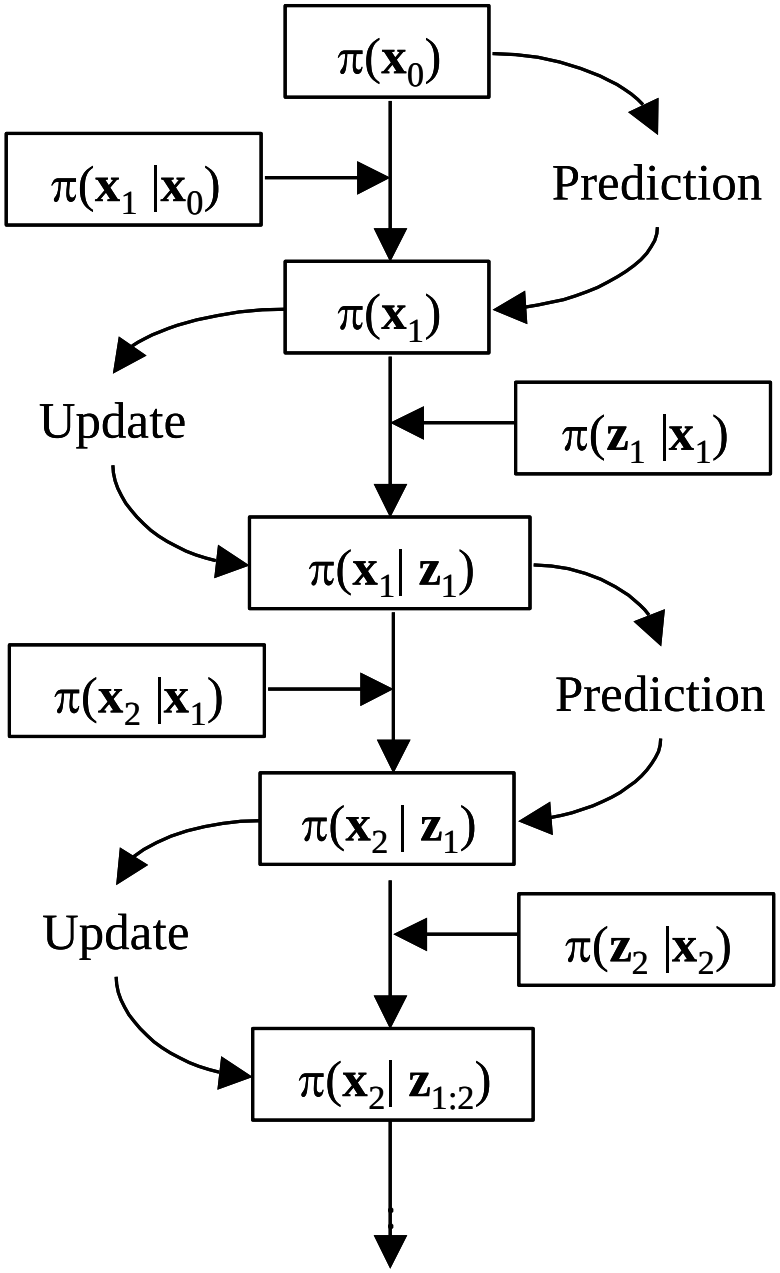
Observation Model: $\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k, \mathbf{n}_k)$

Different problems can be considered:

1. The *prediction problem*, concerned with the determination of $\pi(\mathbf{x}_k \,|\, \mathbf{z}_{1:k-1})$;
2. The *filtering problem*, concerned with the determination of $\pi(\mathbf{x}_k \,|\, \mathbf{z}_{1:k})$;
3. The *fixed-lag smoothing problem*, concerned the determination of $\pi(\mathbf{x}_k \,|\, \mathbf{z}_{1:k+p})$, where $p \geq 1$ is the fixed lag;
4. The *whole-domain smoothing problem*, concerned with the determination of $\pi(\mathbf{x}_k \,|\, \mathbf{z}_{1:K})$, where $\mathbf{z}_{1:K} = \{\mathbf{z}_i, i = 1, \ldots, K\}$ is the complete sequence of measurements.

FILTERING PROBLEM

By assuming that $\pi(\mathbf{x}_0 \,|\, \mathbf{z}_0) = \pi(\mathbf{x}_0)$ is available, the posterior probability density $\pi(\mathbf{x}_k \,|\, \mathbf{z}_{1:k})$ is then obtained with Bayesian filters in two steps: **prediction and update**


$\pi(\mathbf{x}_0)$
$\pi(\mathbf{x}_1 | \mathbf{x}_0)$
Prediction
$\pi(\mathbf{x}_1)$
Update
$\pi(\mathbf{z}_1 | \mathbf{x}_1)$
$\pi(\mathbf{x}_1 | \mathbf{z}_1)$
$\pi(\mathbf{x}_2 | \mathbf{x}_1)$
Prediction
$\pi(\mathbf{x}_2 | \mathbf{z}_1)$
Update
$\pi(\mathbf{z}_2 | \mathbf{x}_2)$
$\pi(\mathbf{x}_2 | \mathbf{z}_{1:2})$

THE KALMAN FILTER

- Evolution and observation models are linear.
- Noises in such models are additive and Gaussian, with known means and covariances.
- Optimal solution if these hypotheses hold.

State Evolution Model: $\mathbf{x}_k = \mathbf{F}_k \mathbf{x}_{k-1} + \mathbf{G}_k \mathbf{u}_{k-1} + \mathbf{s}_{k-1} + \mathbf{v}_{k-1}$

Observation Model: $\mathbf{z}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{n}_k$

- **F** and **H** are known matrices for the linear evolutions of the state **x** and of the observation **z**, respectively.
- **G** is matrix that determines how the control **u** affects the state **x**.
- Vector **s** is assumed to be a known input .
- Noises **v** and **n** have zero means and covariance matrices **Q** and **R**, respectively.

THE KALMAN FILTER

Prediction:

$$\mathbf{x}_k^- = \mathbf{F}_k \mathbf{x}_{k-1} + \mathbf{G}_k \mathbf{u}_{k-1} + \mathbf{s}_{k-1} + \mathbf{v}_{k-1}$$

$$\mathbf{P}_k^- = \mathbf{F}_k \mathbf{P}_{k-1} \mathbf{F}_k^T + \mathbf{Q}_k$$

Update:

$$\mathbf{K}_k = \mathbf{P}_k^- \mathbf{H}_k^T \left(\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k \right)^{-1}$$

$$\mathbf{x}_k = \mathbf{x}_k^- + \mathbf{K}_k (\mathbf{z}_k - \mathbf{H}_k \mathbf{x}_k^-)$$

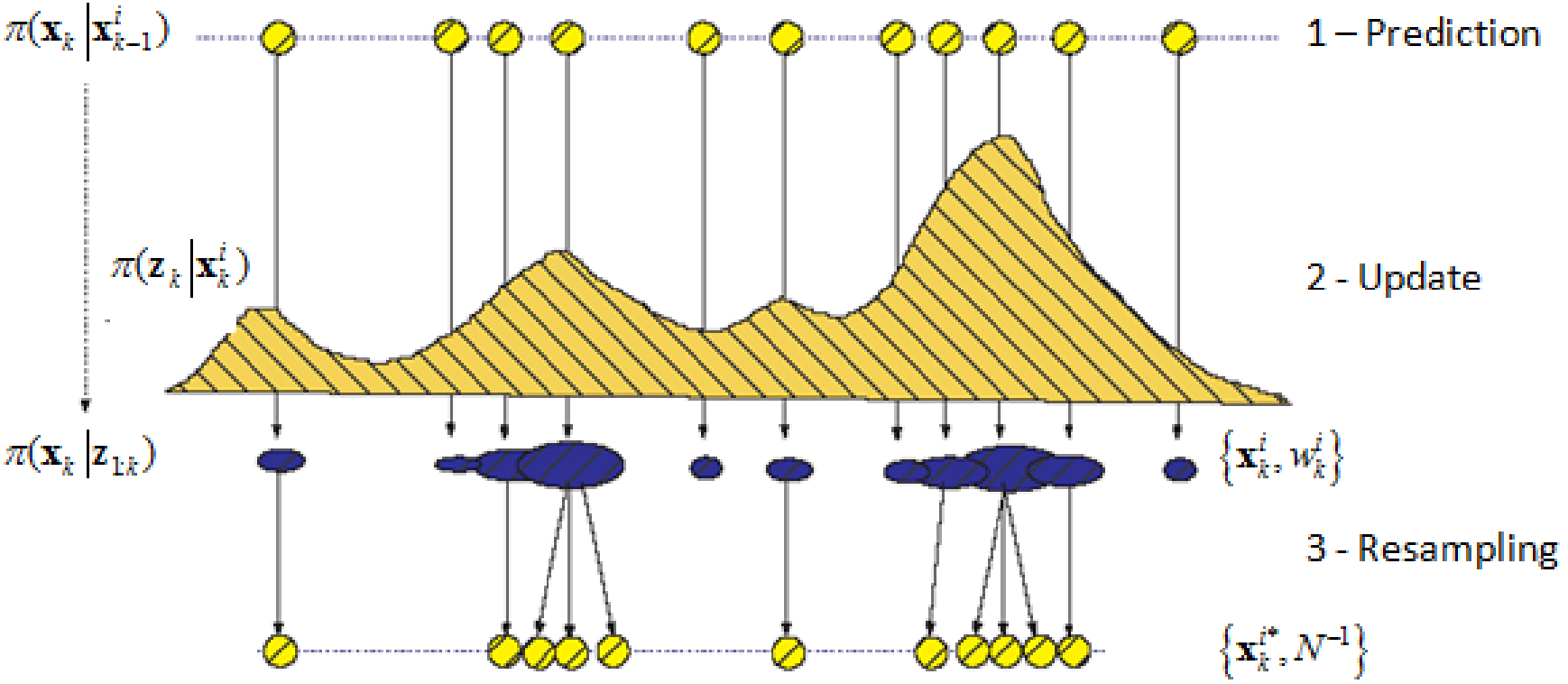
$$\mathbf{P}_k = \left(\mathbf{I} - \mathbf{K}_k \mathbf{H}_k \right) \mathbf{P}_k^-$$

THE PARTICLE FILTER

• Monte-Carlo techniques are the most general and robust for non-linear and/or non-Gaussian distributions.
• The key idea is to represent the required posterior density function by a set of random samples (particles) with associated weights, and to compute the estimates based on these samples and weights.
• Introduced in the 50's, but no much used until recently because of limited computational resources.
• Particles degenerated very fast in early implementations, i.e., most of the particles would have negligible weight. The resampling step has a fundamental role in the advancement of the particle filter.

Sampling Importance Resampling (SIR) Algorithm

(Ristic, B., Arulampalam, S., Gordon, N., 2004, *Beyond the Kalman Filter*, Artech House, Boston)



Sampling Importance Resampling (SIR) Algorithm

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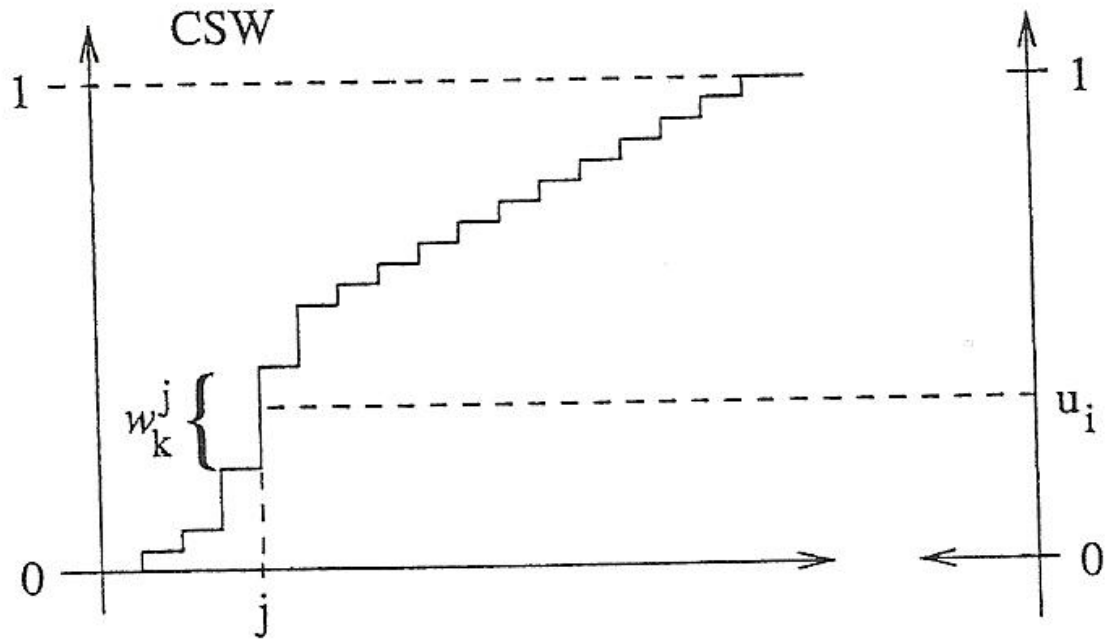


Figure 3.1 The process of resampling: $u_i \sim \mathcal{U}[0, 1]$ maps into index j; the corresponding particle x_k^j has a good chance of being selected and multiplied because of its high value of w_k^j.

Sampling Importance Resampling (SIR) Algorithm

(Ristic, B., Arulampalam, S., Gordon, N., 2004, *Beyond the Kalman Filter*, Artech House, Boston)

Although the resampling step reduces the effects of degeneracy, it introduces other practical problems:

- Limitation in the parallelization.
- Particles that have high weights are statistically selected many times: Loss of diversity, known as sample impoverishment, specially if the evolution model errors are small.

Sampling Importance Resampling (SIR) Algorithm

(Ristic, B., Arulampalam, S., Gordon, N., 2004, *Beyond the Kalman Filter*, Artech House, Boston)

Step 1

For $i=1,\cdots,N$ draw new particles $\mathbf{x}_k^i$ from the prior density $\pi\left(\mathbf{x}_k \left| \mathbf{x}_{k-1}^i \right.\right)$ and then use the likelihood density to calculate the correspondent weights $w_k^i = \pi\left(\mathbf{z}_k \left| \mathbf{x}_k^i \right.\right)$.

Step 2

Calculate the total weight $T_w = \sum_{i=1}^{N} w_k^i$ and then normalize the particle weights, that is, for $i=1,\cdots,N$ let $w_k^i = T_w^{-1} w_k^i$

Step 3

Resample the particles as follows :

Construct the cumulative sum of weights (CSW) by computing $c_i = c_{i-1} + w_k^i$ for $i=1,\cdots,N$, with $c_0 = 0$.

Let $i=1$ and draw a starting point u_1 from the uniform distribution $U\left[0, N^{-1}\right]$

For $j=1,\cdots,N$

Move along the CSW by making $u_j = u_1 + N^{-1}(j-1)$

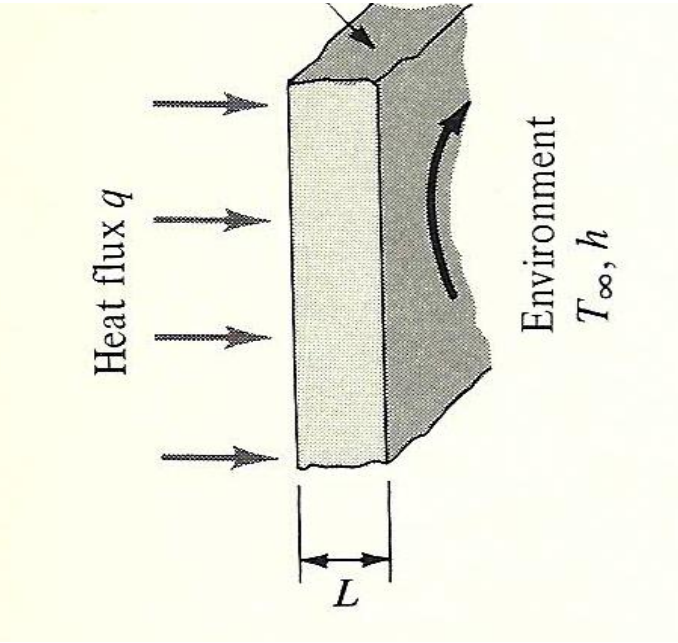
While $u_j > c_i$ make $i = i+1$.

Assign sample $x_k^j = x_k^i$

Assign sample $w_k^j = N^{-1}$

- Weights are easily evaluated and importance density easily sampled.
- Sampling of the importance density is independent of the measurements at that time. The filter can be sensitive to outliers.
- Resampling is applied every iteration, which can result in fast loss of diversity of the particles.

EXAMPLE: Lumped System



$$\frac{d\theta(t)}{dt} + m\theta(t) = \frac{mq(t)}{h} \qquad \text{for } t > 0$$

$$\theta = \theta_0 \qquad \text{for } t = 0$$

where

$$\theta(t) = T(t) - T_\infty$$

$$\theta_0 = T_0 - T_\infty$$

$$m = \frac{h}{\rho c L}$$

EXAMPLE: Lumped System

Two illustrative cases are examined:

(i) Heat Flux $q(t) = q_0$ constant and deterministically known;
(ii) Heat Flux $q(t) = q_0 f(t)$ with unknown time variation.

• **Plate is made of aluminum** ($\rho = 2707$ kgm^{-3}, $c = 896$ Jkg^{-1}K^{-1}), with thickness $L = 0.03$ m, $q_0 = 8000$ Wm^{-2}, $=20$ ^{o}C, $h = 50$ Wm^{-2}K^{-1} and $T_0 = 50$ ^{o}C.
• **Measurements of the transient temperature of the slab are assumed available**. These measurements contain additive, uncorrelated, Gaussian errors, with zero mean and a constant standard deviation σ_z.
• The **errors in the state evolution model** are also supposed to be additive, uncorrelated, Gaussian, with zero mean and a constant standard deviation σ_θ.

(i) Heat Flux $q(t) = q_0$ constant and deterministically known

The analytical solution for this problem is given by:

$$\theta(t) = \theta_0 e^{-mt} + \frac{q_0}{h}(1 - e^{-mt}) \tag{A.3}$$

The only state variable in this case is the temperature $\theta(t_k) = \theta_k$ since the applied heat flux q_0 is constant and deterministically known, as the other parameters appearing in the formulation. By using a forward finite-differences approximation for the time derivative in equation (A.1.a), we obtain:

$$\theta_k = (1 - m\Delta t)\theta_{k-1} + \frac{mq_0}{h}\Delta t \tag{A.4}$$

Therefore, the state and observation models given by equations (3.a,b) are obtained with:

$$\mathbf{x}_k = [\theta_k] \qquad \mathbf{F}_k = [(1 - m\Delta t)] \quad \mathbf{s}_k = \left[m\frac{q_0}{h}\Delta t \right] \quad \mathbf{H}_k = [1] \qquad \mathbf{Q}_k = [\sigma_\theta^2] \qquad \mathbf{R}_k = [\sigma_z^2] \tag{A5.a-f}$$

$$\mathbf{x}_k = \mathbf{F}_k \mathbf{x}_{k-1} + \mathbf{G}_k \mathbf{u}_{k-1} + \mathbf{s}_{k-1} + \mathbf{v}_{k-1}$$

$$\mathbf{z}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{n}_k$$

(ii) Heat Flux $q(t) = q_0 f(t)$ with unknown time variation

The analytical solution for this problem is given by:

$$\theta(t) = e^{-mt}\left\{\theta_0 + \frac{mq_0}{h}\int_{t'=0}^{t} e^{mt'} f(t')\,dt'\right\} \tag{A.6}$$

In this case, the state variables are given by the temperature $\theta(t_k) = \theta_k$ and the function that gives the time variation of the applied heat flux, that is, $f(t_k) = f_k$ .As in the case examined above, the applied heat flux q_0 is constant and deterministically known, as the other parameters appearing in the formulation. By using a forward finite-differences approximation for the time derivative in equation (A.1.a), we obtain the equation for the evolution of the state variable $\theta(t_k) = \theta_k$:

$$\theta_k = (1 - m\Delta t)\theta_{k-1} + \left(\frac{mq_0}{h}\Delta t\right) f_{k-1} \tag{A.7}$$

A random walk model is used for the state variable $f(t_k) = f_k$, which is given in the form:

$$f_k = f_{k-1} + \varepsilon_{k-1} \tag{A.8}$$

where ε_{k-1} is Gaussian with zero mean and constant standard deviation σ_{rw}.

(ii) Heat Flux $q(t) = q_0 f(t)$ with unknown time variation

Therefore, the state and observation models given by equations (3.a,b) are obtained with:

$$\mathbf{x}_k = \begin{bmatrix} \theta_k \\ f_k \end{bmatrix} \qquad \mathbf{F}_k = \begin{bmatrix} (1 - m\Delta t) & \dfrac{mq_0}{h}\Delta t \\ 0 & 1 \end{bmatrix} \qquad \mathbf{s}_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(A.9.a-c)

$$\mathbf{H}_k = [1 \;\; 0] \qquad \mathbf{Q}_k = \begin{bmatrix} \sigma_\theta^2 & 0 \\ 0 & \sigma_{rw}^2 \end{bmatrix} \qquad \mathbf{R}_k = [\sigma_z^2]$$

(A.9.d-f)

$$\mathbf{x}_k = \mathbf{F}_k \mathbf{x}_{k-1} + \mathbf{G}_k \mathbf{u}_{k-1} + \mathbf{s}_{k-1} + \mathbf{v}_{k-1}$$

$$\mathbf{z}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{n}_k$$

EXAMPLE: 1D Heat conduction

Linear Heat Conduction Problem

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \quad \text{in } 0 < x < L, \text{ for } t > 0$$

$$T = 0 \quad \text{at } x=0, \text{ for } t > 0$$

$$T = T^* \quad \text{at } x=L, \text{ for } t > 0$$

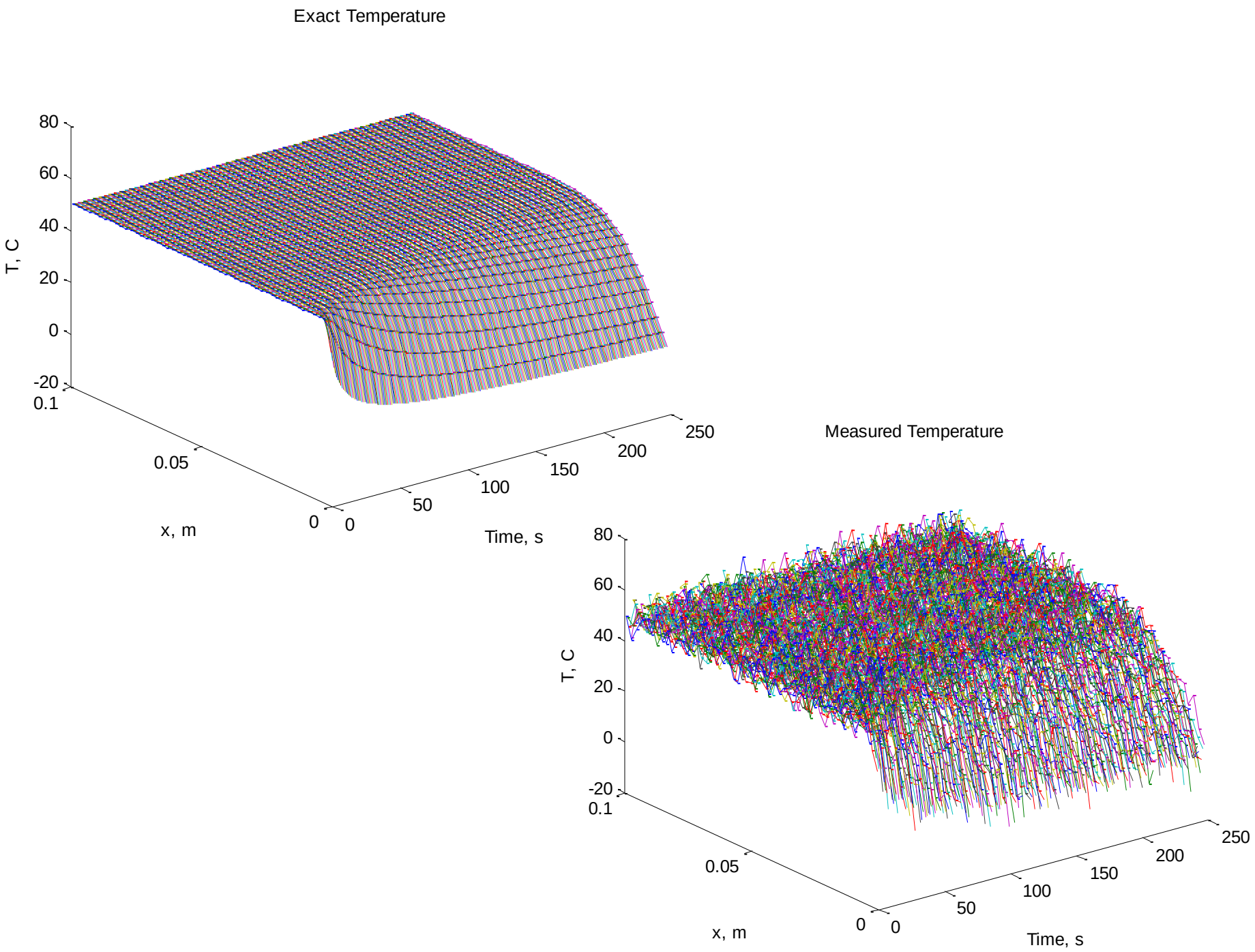
$$T = T^* \quad \text{for } t=0, \text{ in } 0 < x < L$$

Explicit finite-differences: $\mathbf{T}^{k+1} = \mathbf{F}\mathbf{T}^{k} + \mathbf{S}$

$$\mathbf{T} = \begin{bmatrix} T_1 \\ \vdots \\ T_N \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} (1-2r) & r & & & \\ r & (1-2r) & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & (1-2r) & r \\ & & & r & (1-2r) \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ rT^* \end{bmatrix}$$

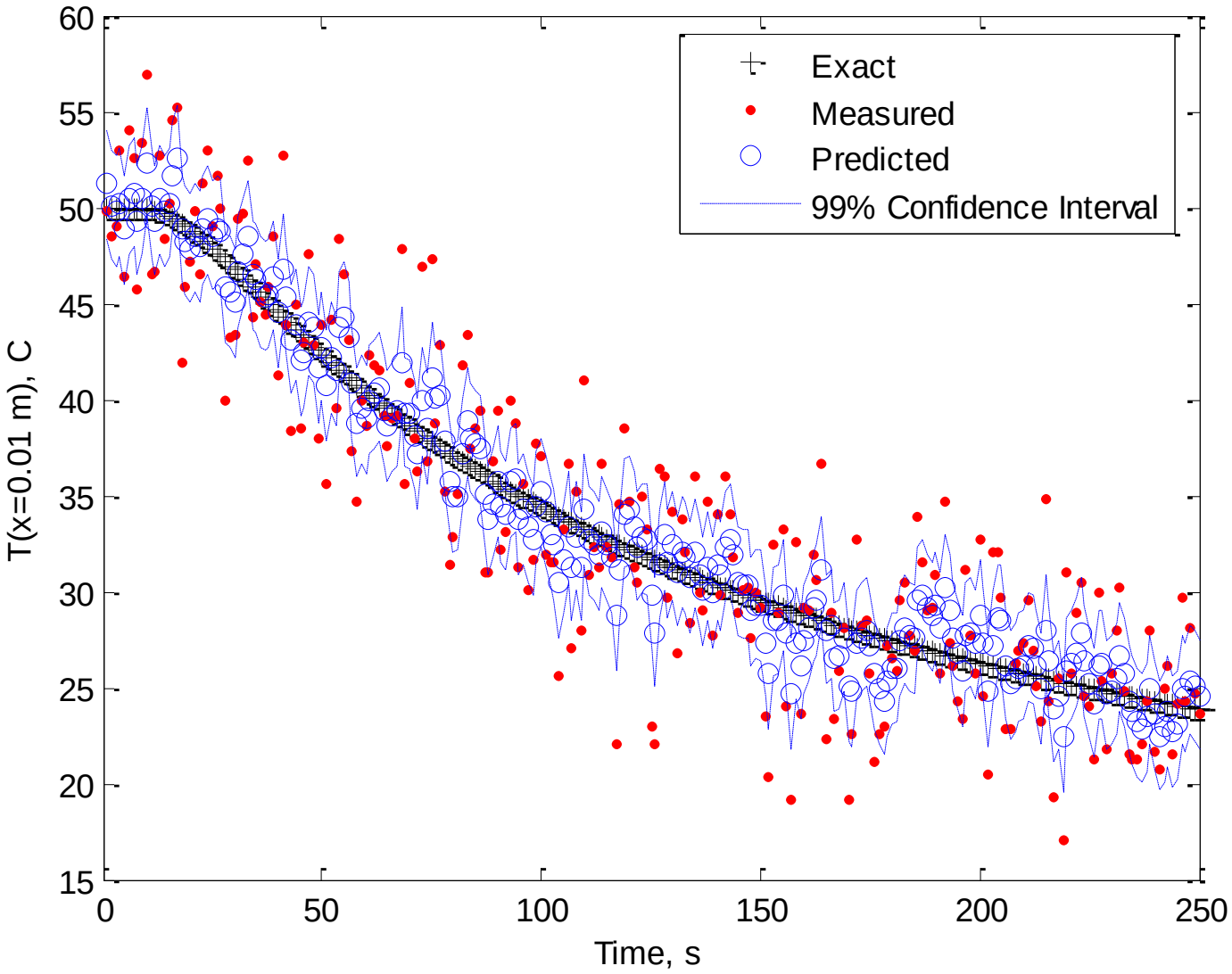
EXAMPLE: 1D Heat conduction

- Concrete with thermal diffusivity $\alpha = 4.9 \times 10^{-7}$ m^2/s
- Standard-deviation for the measurement errors = 2 ^oC
- Final time = 250 seconds
- Measurements available in the region every 1 second
- L = 0.1 m
- N = 50 internal nodes


Exact Temperature
T, C
80
60
40
20
0
-20
0.1
0.05
0
x, m
0
50
100
150
200
250
Time, s
Measured Temperature
T, C
80
60
40
20
0
-20
0.1
0.05
0
x, m
0
50
100
150
200
250
Time, s

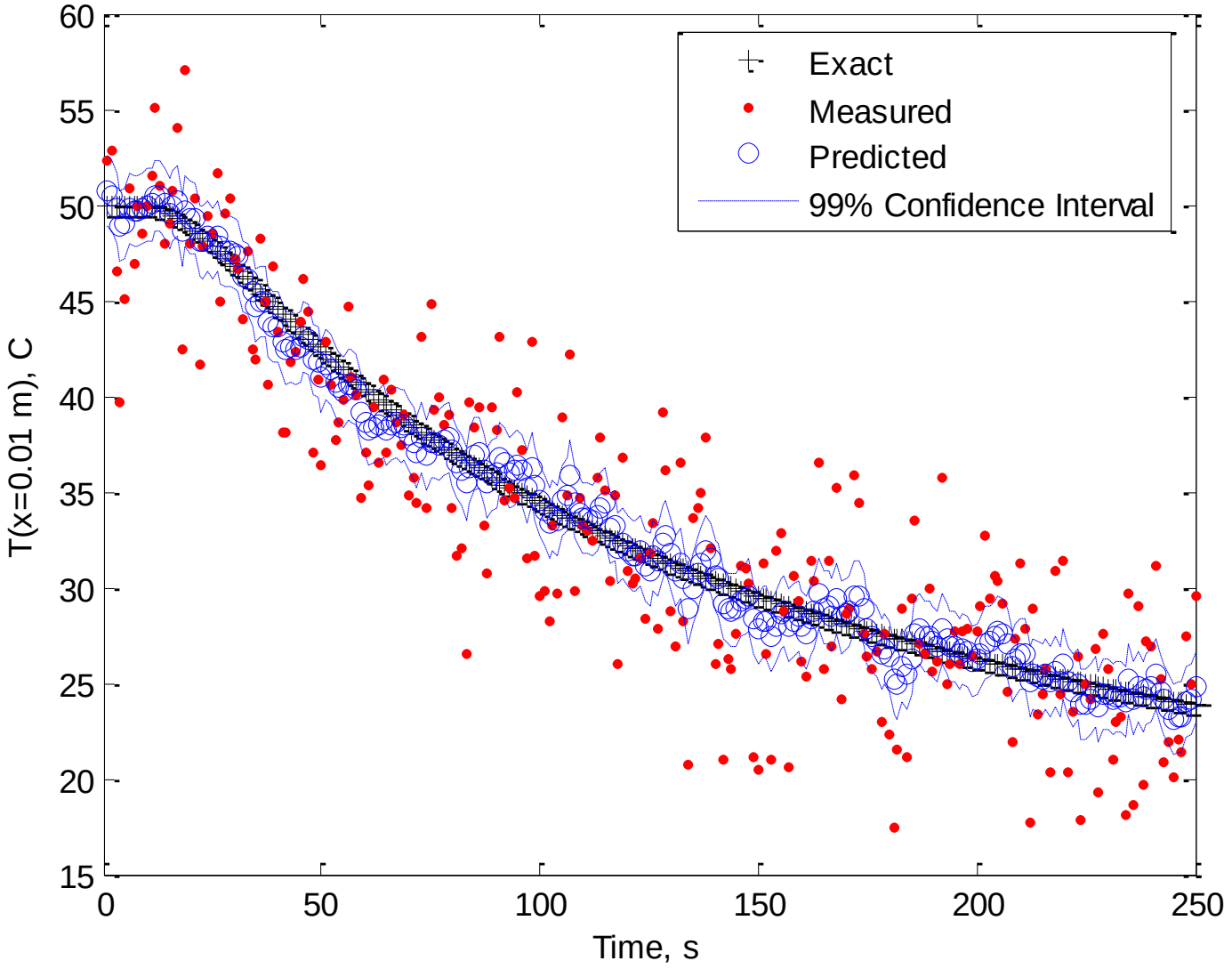
Kalman Filter

(Standard-deviation for the evolution model errors of 1 ºC)



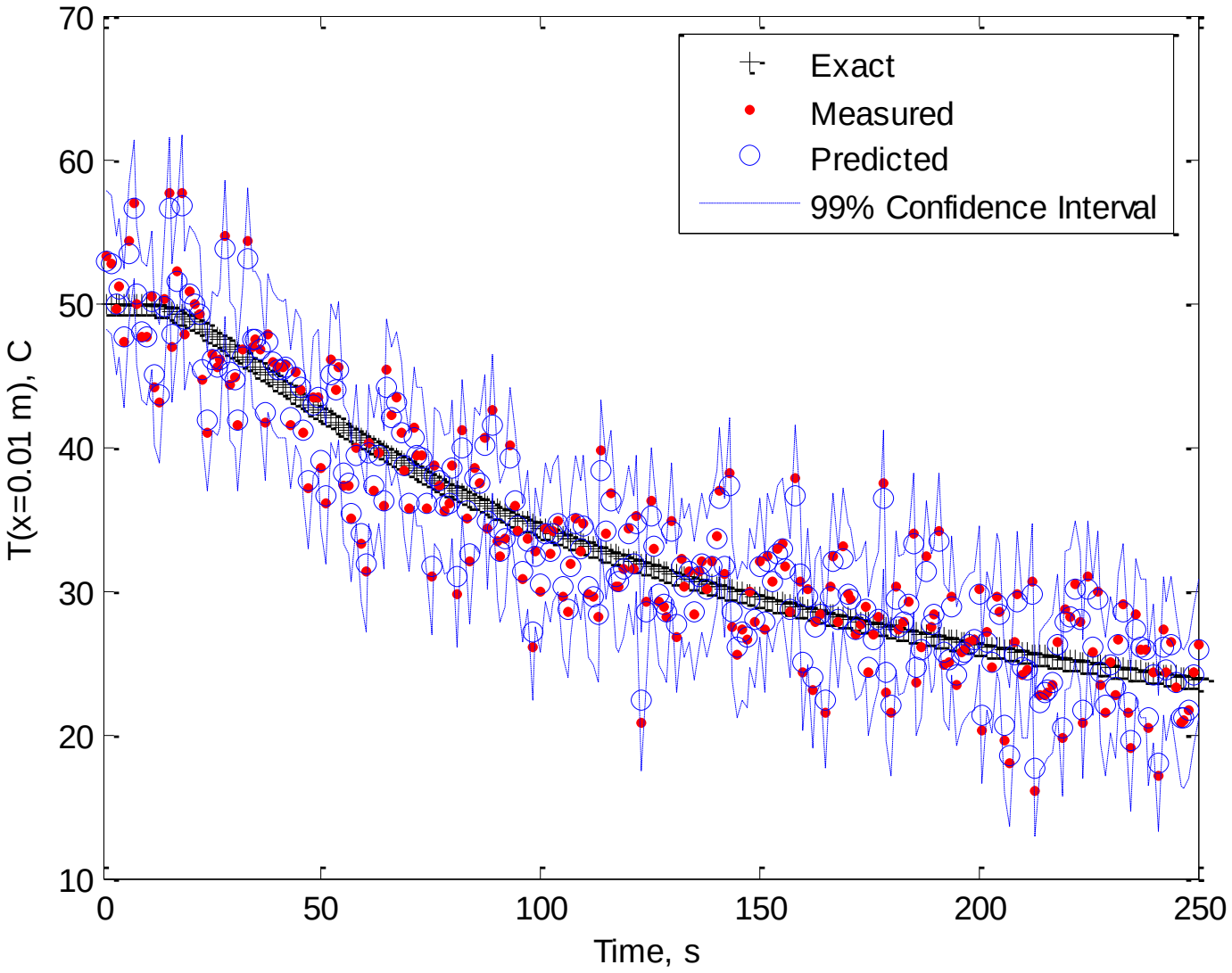
Kalman Filter

(Standard-deviation for the evolution model errors of 0.5 ^{o}C)



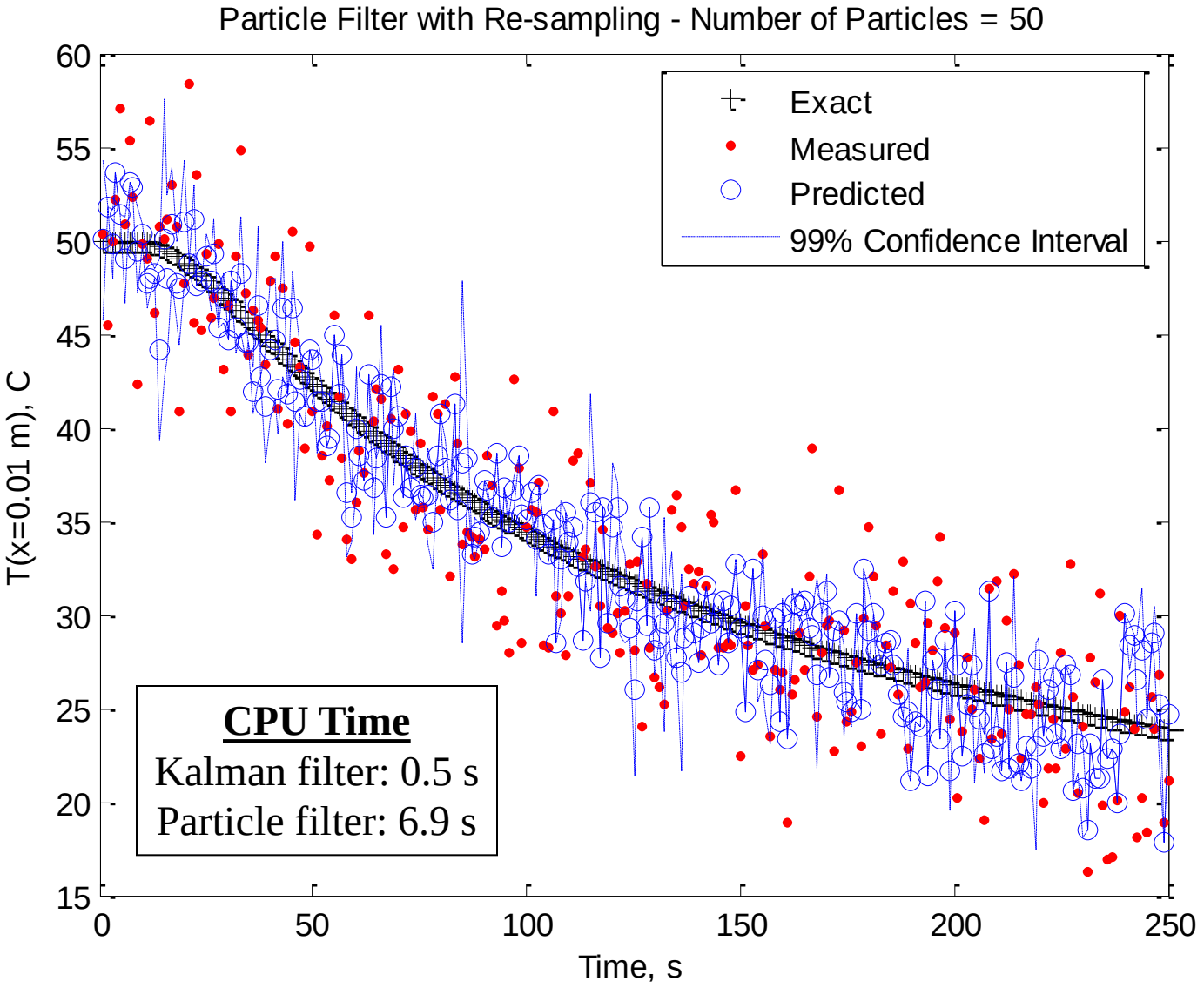
Kalman Filter

(Standard-deviation for the evolution model errors of 5 ºC)



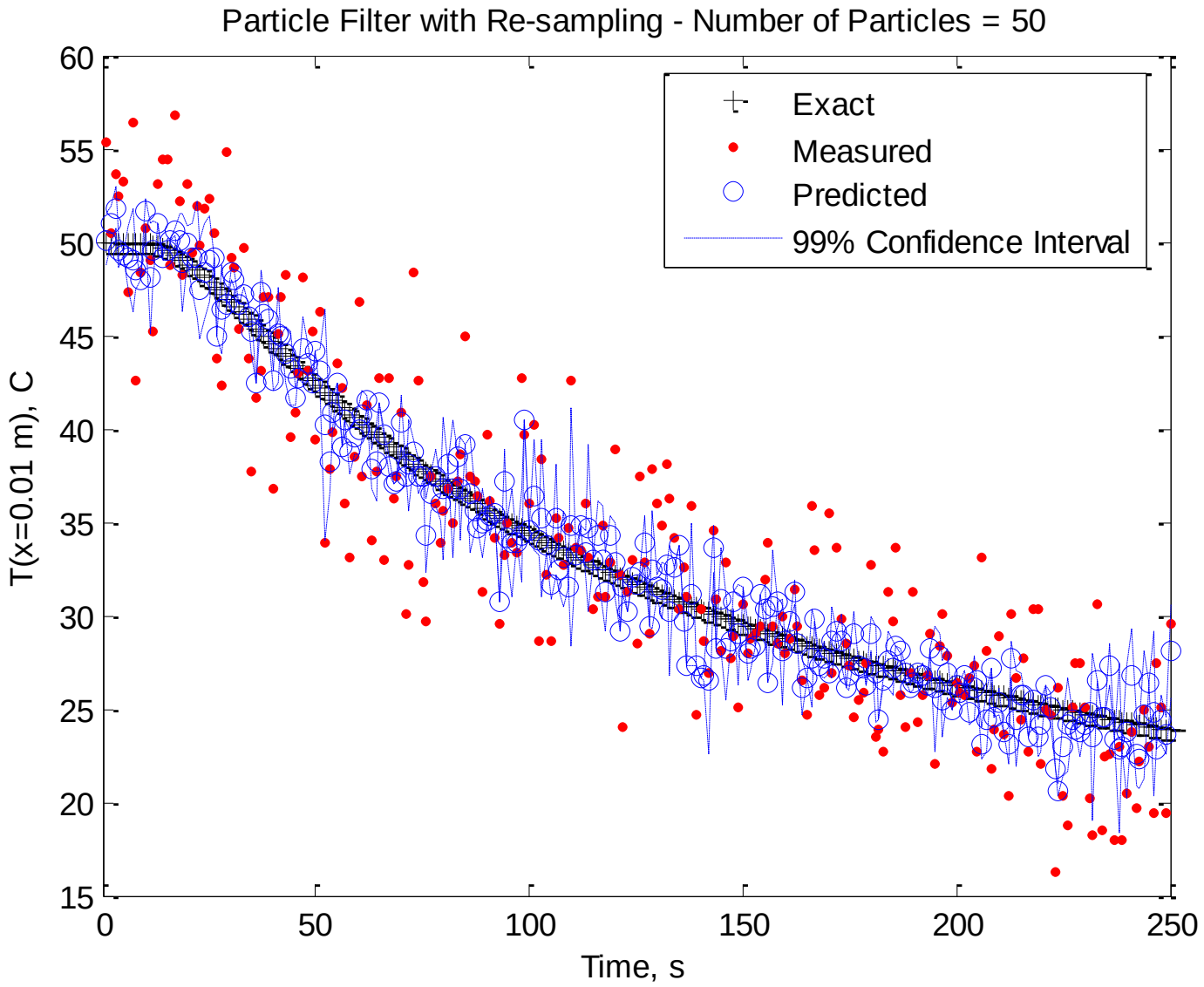
Particle Filter

(Standard-deviation for the evolution model errors of 1 ^{o}C)



Particle Filter

(evolution model errors with uniform distribution in [-1,1] ^{o}C)



EXAMPLE: 1D Heat conduction

Nonlinear Heat Conduction Problem

$$C(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[K(T)\frac{\partial T}{\partial x} \right] \qquad \text{in } 0 < x < L, \text{ for } t > 0$$

$$\frac{\partial T}{\partial x} = 0 \qquad \text{at } x=0, \text{ for } t > 0$$

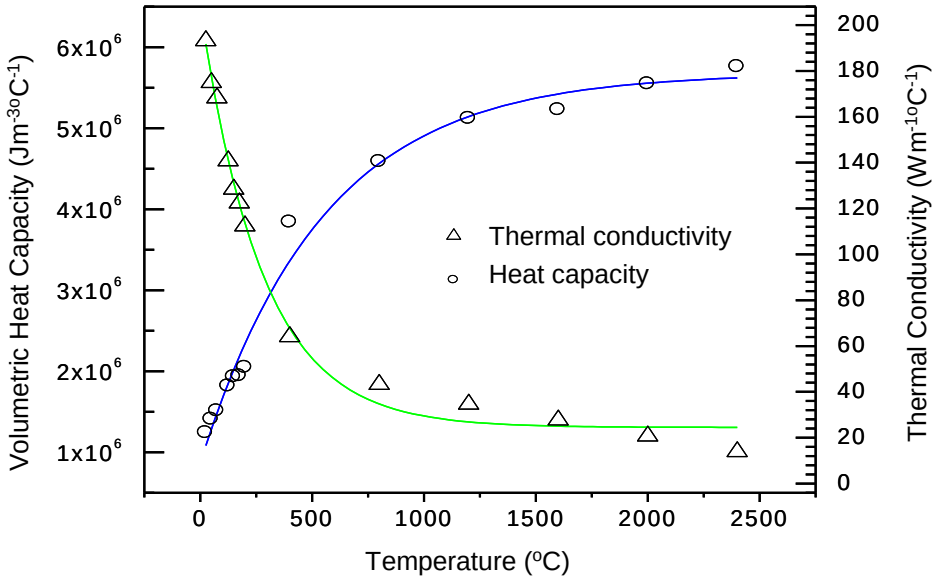
$$k(T)\frac{\partial T}{\partial x} = q^* \qquad \text{at } x=L, \text{ for } t > 0$$

$$T = T^* \qquad \text{for } t=0, \text{ in } 0 < x < L$$

Graphite

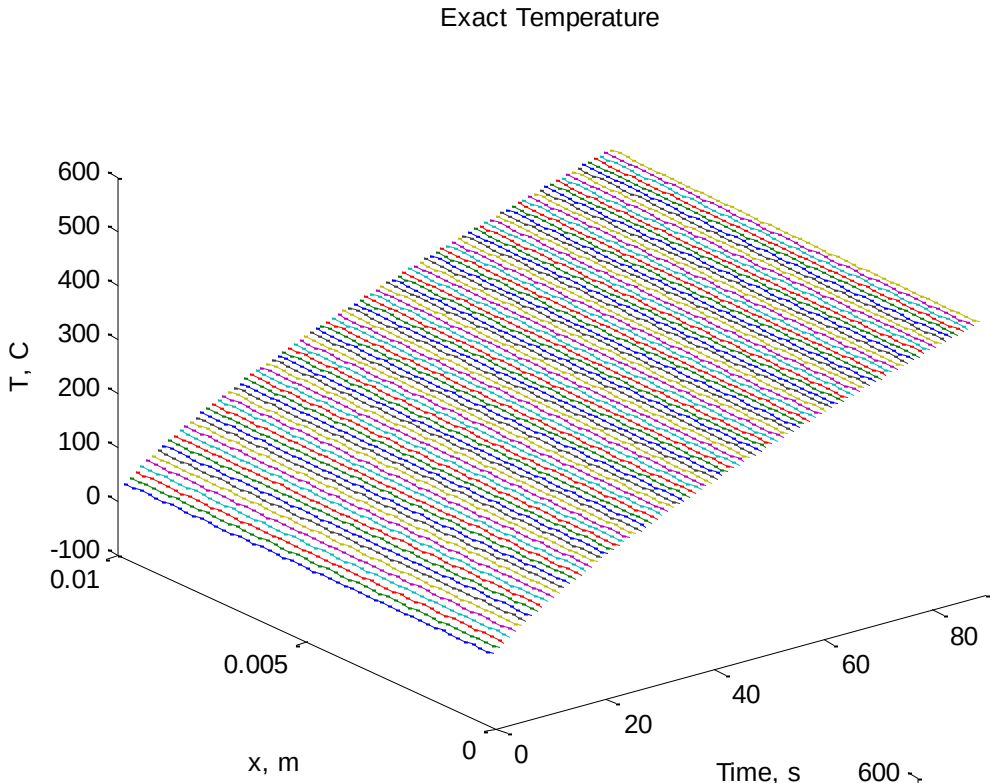
$$C(T) = A_1 + A_2 e^{-T/A_3}$$

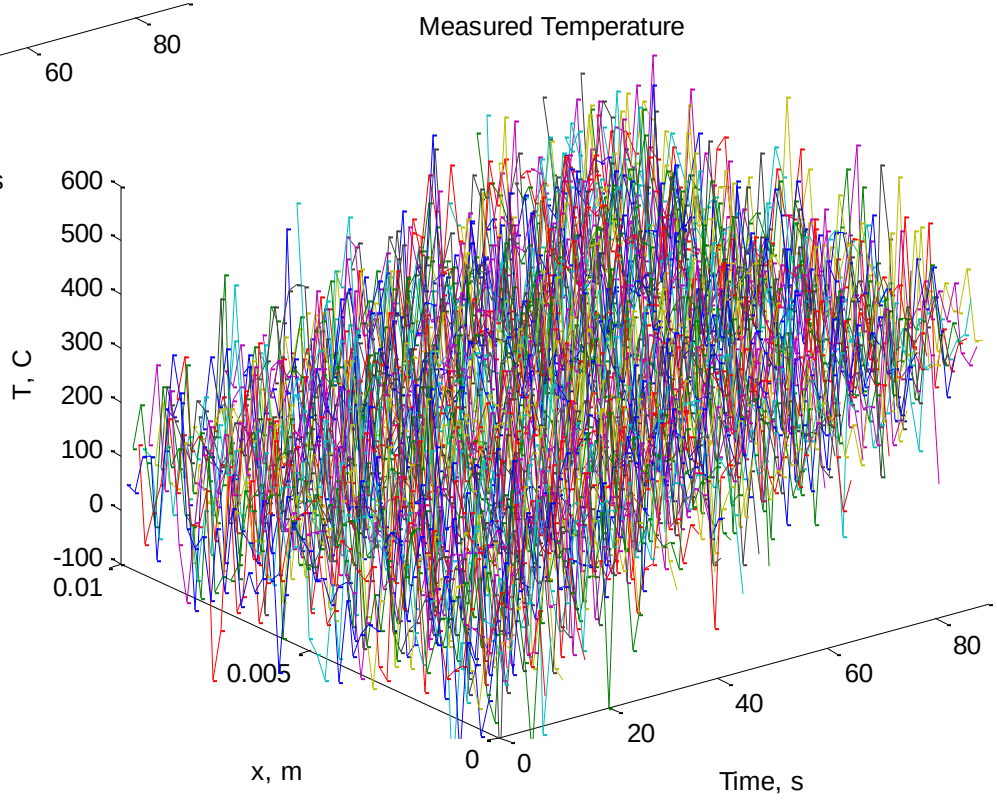
$$k(T) = B_1 + B_2 e^{-T/B_3}$$

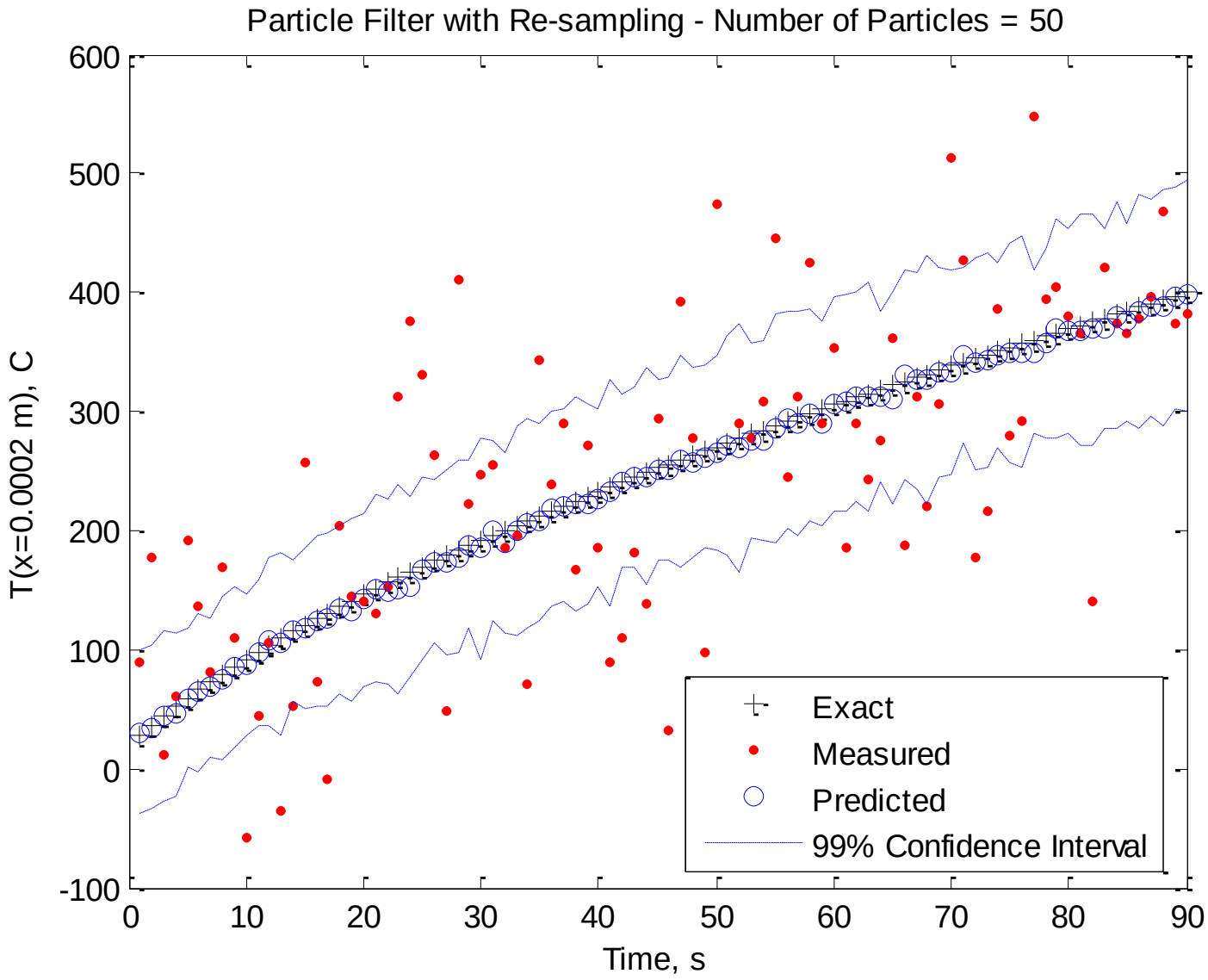


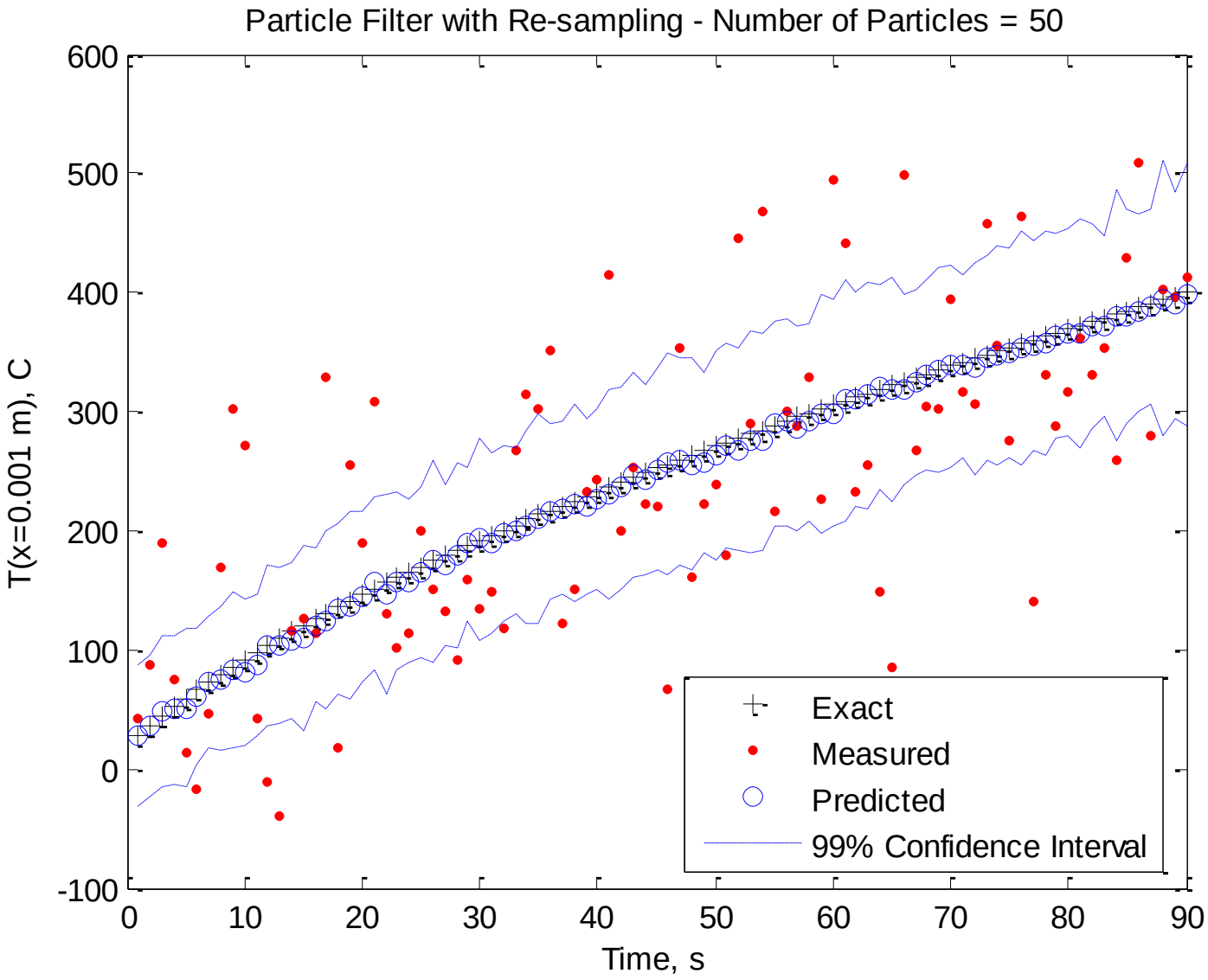
EXAMPLE: 1D Heat conduction

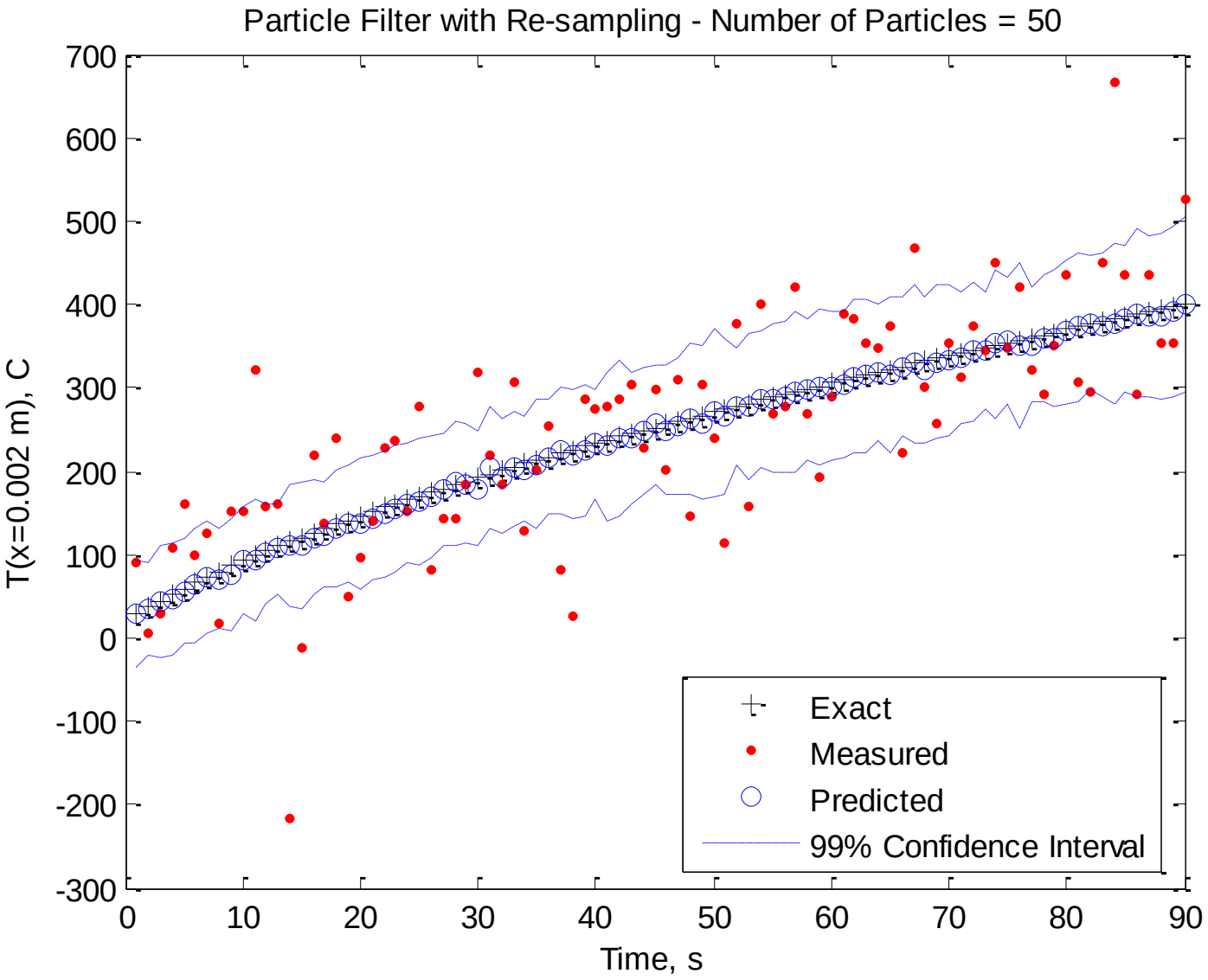
- $T^* = 20\ ^oC$
- $q^* = 10^5\ W/m^2$
- $L = 0.01\ m$
- 50 finite-volumes
- Final time = 90 s
- Time step = 1 s
- Errors in the state evolution and observation models were supposed to be additive, Gaussian, uncorrelated, with zero mean and constant standard-deviations
- Standard-deviation for the state evolution model = 5 ^oC
- Standard-deviation for the observation model = 10 ^oC

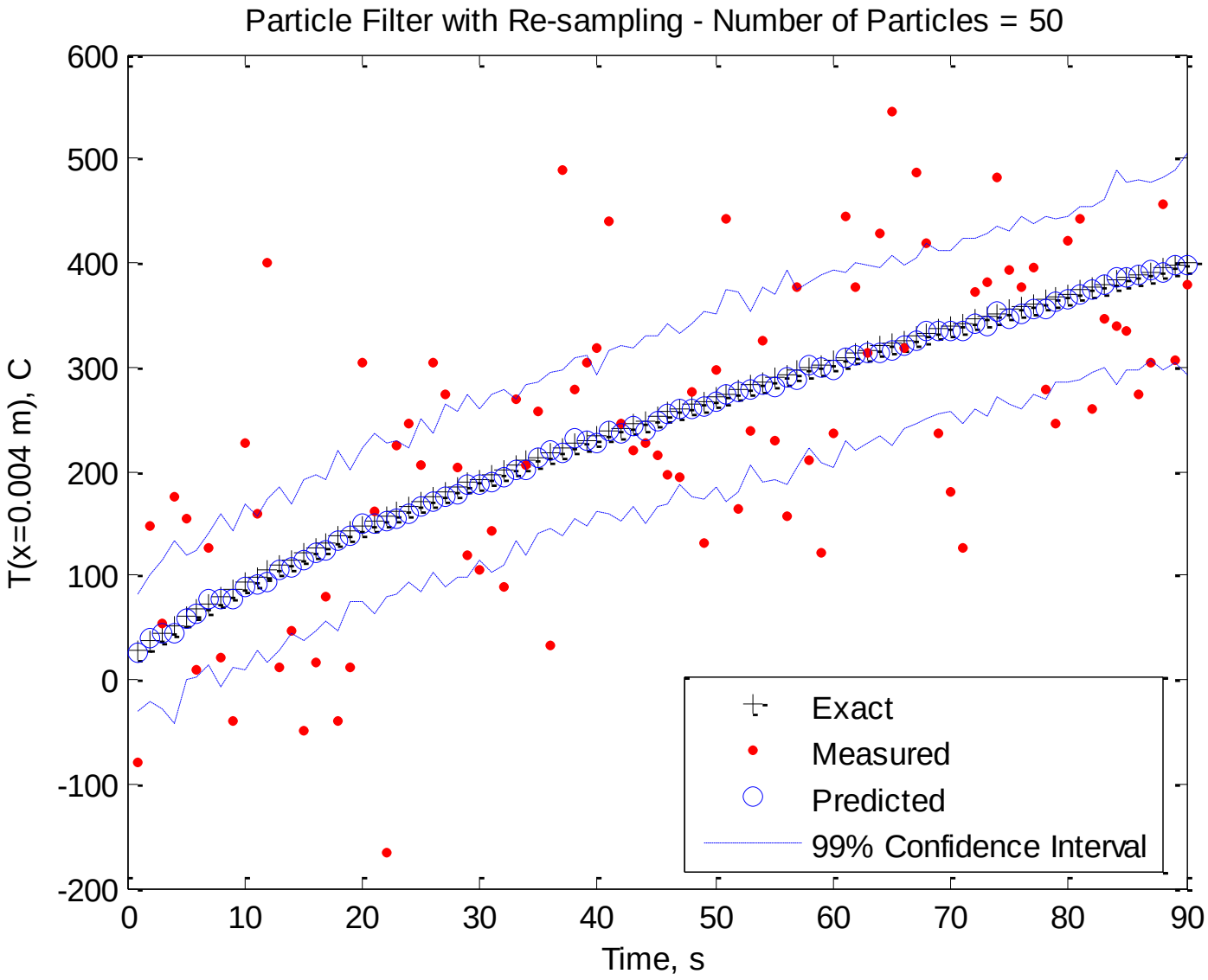

Exact Temperature
T, C
600
500
400
300
200
100
0
-100
0.01
0.005
0
x, m
0
20
40
60
80
Time, s


Measured Temperature
T, C
600
500
400
300
200
100
0
-100
0.01
0.005
0
x, m
0
20
40
60
80
Time, s


Particle Filter with Re-sampling - Number of Particles = 50
T(x=0.0002 m), C
Time, s
600
500
400
300
200
100
0
-100
0
10
20
30
40
50
60
70
80
90
Exact
Measured
Predicted
99% Confidence Interval


Particle Filter with Re-sampling - Number of Particles = 50
T(x=0.001 m), C
Time, s
600
500
400
300
200
100
0
-100
0
10
20
30
40
50
60
70
80
90
Exact
Measured
Predicted
99% Confidence Interval


Particle Filter with Re-sampling - Number of Particles = 50
T(x=0.002 m), C
Time, s
700
600
500
400
300
200
100
0
-100
-200
-300
0
10
20
30
40
50
60
70
80
90
Exact
Measured
Predicted
99% Confidence Interval


Particle Filter with Re-sampling - Number of Particles = 50
T(x=0.004 m), C
Time, s
600
500
400
300
200
100
0
-100
-200
0
10
20
30
40
50
60
70
80
90
Exact
Measured
Predicted
99% Confidence Interval

APPLICATIONS

Estimation of position-dependent transient heat source

$$C\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2} - \frac{h}{e}\left(T - T_\infty\right) + \frac{g(x,y,t)}{e} \quad \text{at } 0 < x < L,\ 0 < y < L, \quad \text{for } t > 0$$

$$\frac{\partial T}{\partial x} = 0 \quad \text{at } x = 0 \text{ and } x = L, \text{ for } t > 0$$

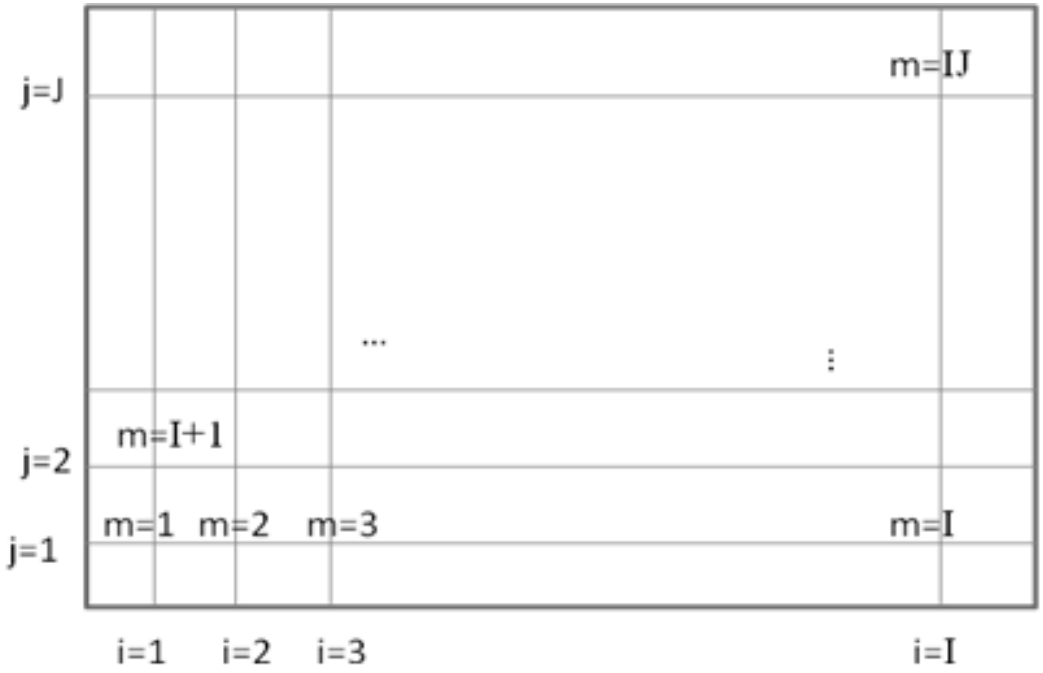
$$\frac{\partial T}{\partial y} = 0 \quad \text{at } y = 0 \text{ and } y = L, \text{ for } t > 0$$

$$T = T_0 \quad \text{for } t = 0, \text{ at } 0 < x < L \text{ and } 0 < y < L \qquad (4)$$

H.Massard, F. sepulveda, H.R.B. Orlande and O. Fudym, 2010, Kalman Filtering for thermal diffusivity and transient source term mapping from infrared images, *Inverse Problems, Design and Optimization Symposium* João Pessoa, Paraíba, Brazil, August 23-27, 2010

APPLICATIONS

Estimation of position-dependent transient heat source



H.Massard, F. sepulveda, H.R.B. Orlande and O. Fudym, 2010, Kalman Filtering for thermal diffusivity and transient source term mapping from infrared images, *Inverse Problems, Design and Optimization Symposium* João Pessoa, Paraíba, Brazil, August 23-27, 2010

APPLICATIONS

Estimation of position-dependent transient heat source

$$\mathbf{T}^{k+1} = \mathbf{T}^{k} + \mathbf{J}^{k}\mathbf{P}^{k} + \mathbf{v}_{T}^{k+1} \qquad \mathbf{P}^{k+1} = \mathbf{I}\mathbf{P}^{k} + \mathbf{v}_{P}^{k}$$

$$\mathbf{J}^{k} = \begin{bmatrix} L_1^k & -\Delta t(T_1^k - T_\infty) & \Delta t & 0 & 0 & & \\ 0 & 0 & 0 & L_2^k & -\Delta t(T_2^k - T_\infty) & \Delta t & 0 \\ \vdots & & & & & & \\ 0 & \cdots & & & \cdots 0 & L_M^k & -\Delta t(T_M^k - T_\infty) \quad \Delta t \end{bmatrix}$$

H.Massard, F. sepulveda, H.R.B. Orlande and O. Fudym, 2010, Kalman Filtering for thermal diffusivity and transient source term mapping from infrared images, *Inverse Problems, Design and Optimization Symposium* João Pessoa, Paraíba, Brazil, August 23-27, 2010

APPLICATIONS

Estimation of position-dependent transient heat source

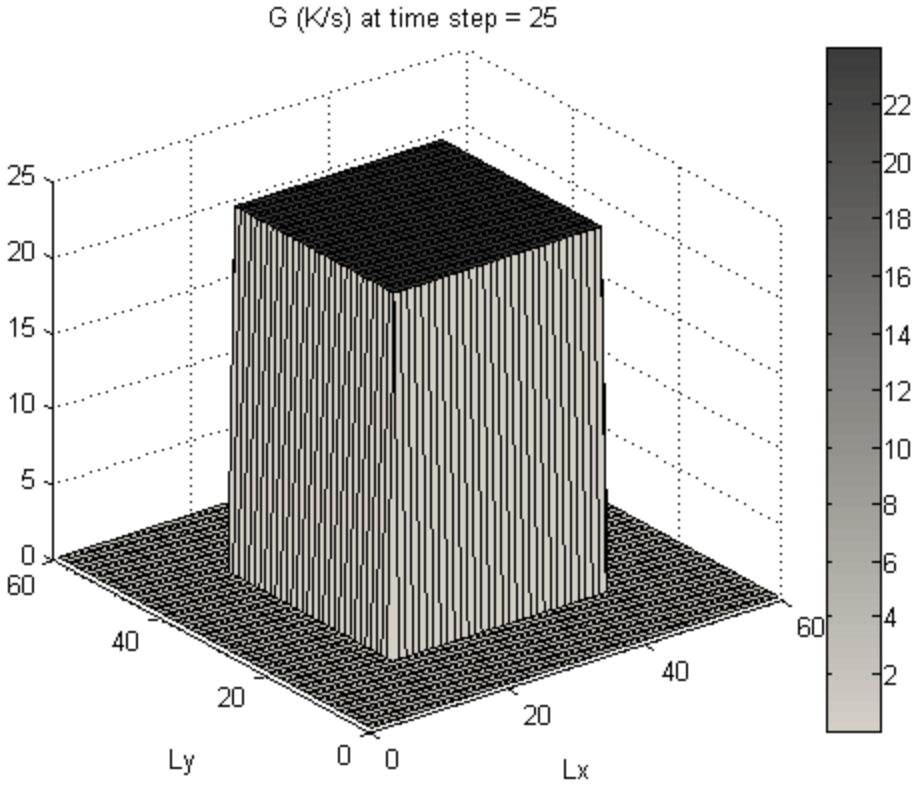
$$\mathbf{T}^{k+1} = \mathbf{T}^{k} + \mathbf{J}^{k}\mathbf{P}^{k} + \mathbf{v}_{T}^{k+1} \qquad \mathbf{P}^{k+1} = \mathbf{I}\mathbf{P}^{k} + \mathbf{v}_{P}^{k}$$

$$\mathbf{T}^{k} = \begin{bmatrix} T_1^k \\ T_2^k \\ \vdots \\ T_M^k \end{bmatrix} \qquad \mathbf{P}^{k} = \begin{bmatrix} \begin{bmatrix} a_1^k \\ H_1^k \\ G_1^k \end{bmatrix} \\ \begin{bmatrix} a_2^k \\ H_2^k \\ G_2^k \end{bmatrix} \\ \vdots \\ \begin{bmatrix} a_M^k \\ H_M^k \\ G_M^k \end{bmatrix} \end{bmatrix}$$

H.Massard, F. sepulveda, H.R.B. Orlande and O. Fudym, 2010, Kalman Filtering for thermal diffusivity and transient source term mapping from infrared images, *Inverse Problems, Design and Optimization Symposium*, João Pessoa, Paraíba, Brazil, August 23-27, 2010

APPLICATIONS

Estimation of position-dependent transient heat source



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APPLICATIONS

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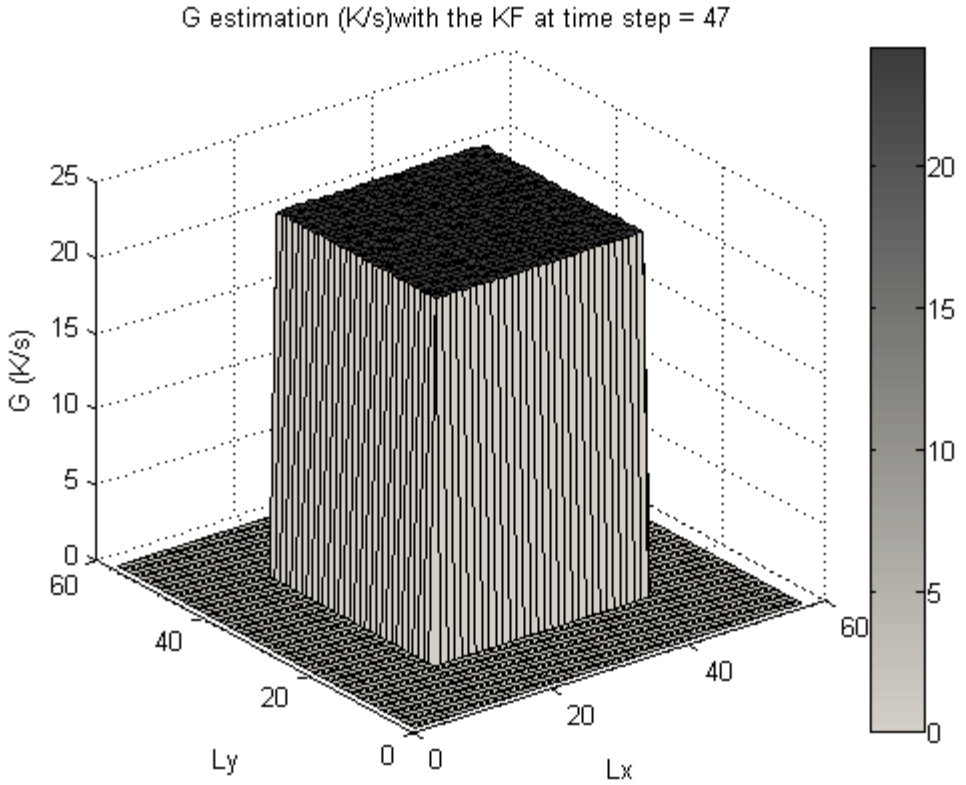

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APPLICATIONS

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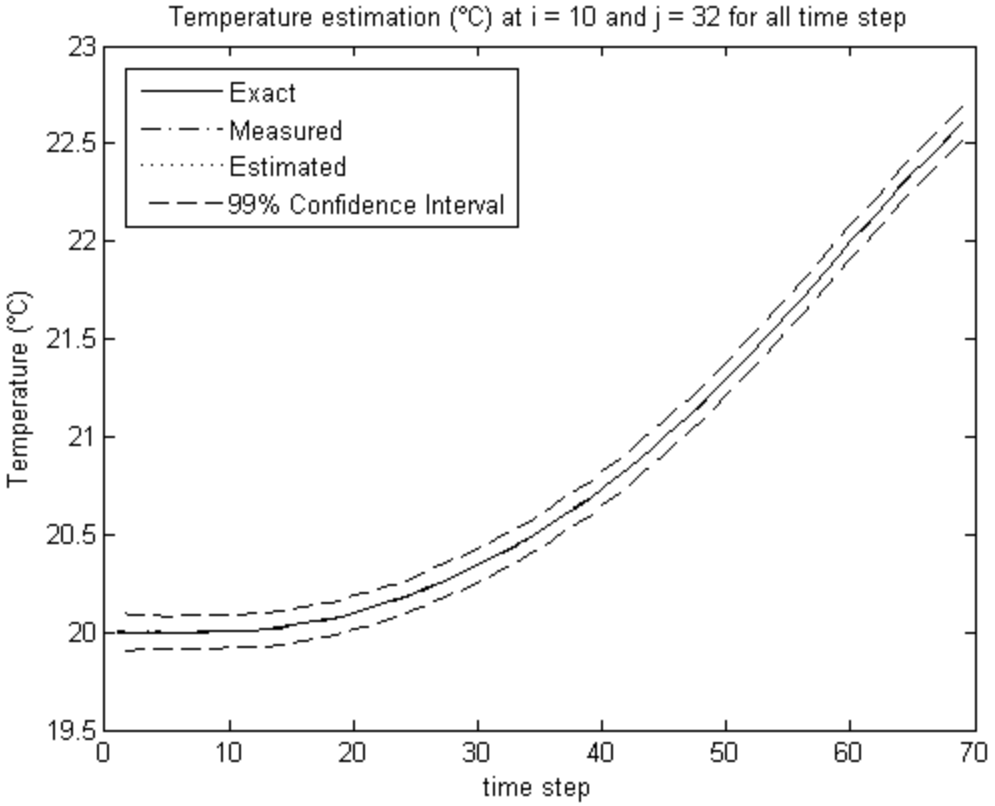

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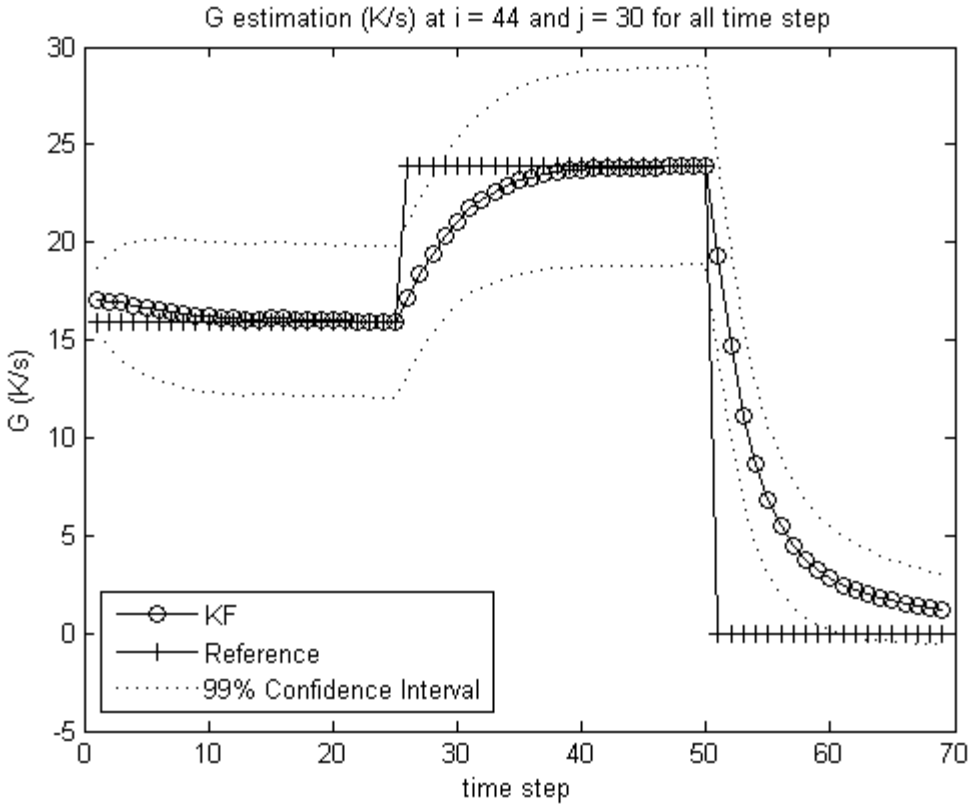

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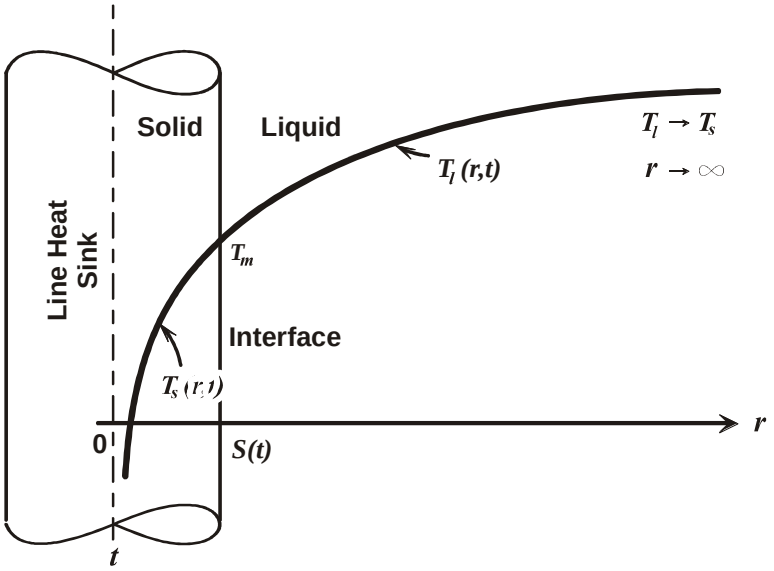
APPLICATIONS

Estimation of the location of the solidification front and the intensity of a line heat sink



Silva, W. B. Orlande, H. R. B.; Colaço, M. J., Fudym, O., 2011, Application Of Bayesian Filters To A One-Dimensional Solidification Problem, *21st Brazilian Congress of Mechanical Engineering*, Natal, RN, Brazil.

APPLICATIONS

The mathematical formulation for the solid phase is given as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_s(r,t)}{\partial r}\right)=\frac{1}{\alpha_s}\frac{\partial T_s(r,t)}{\partial t} \quad in \quad 0<r<S(t) \quad and \quad t>0 \tag{1.a}$$

while the liquid phase is described as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_l(r,t)}{\partial r}\right)=\frac{1}{\alpha_l}\frac{\partial T_l(r,t)}{\partial t} \quad in \quad S(t)<r<\infty \quad and \quad t>0 \tag{1.b}$$

$$T_l(r,t)\to T_i \quad in \quad r\to\infty \quad and \quad t>0 \tag{1.c}$$

$$T_l(r,t)=T_i \quad in \quad t=0 \quad and \quad r>0 \tag{1.d}$$

At the interface between liquid and solid phases, the following conditions must be satisfied

$$T_s(r,t)=T_l(r,t)=T_m \quad in \quad r=S(t) \quad and \quad t>0 \tag{1.e}$$

$$k_s\frac{\partial T_s(r,t)}{\partial r}-k_l\frac{\partial T_l(r,t)}{\partial r}=\rho L\frac{\partial S(t)}{\partial t} \quad in \quad r=S(t) \quad and \quad t>0 \tag{1.f}$$

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APPLICATIONS

An analytical solution of this problem can be obtained for this physical problem and it is given by [8]:

$$T_s(r,t) = T_m + \frac{Q_s}{4\pi k_s}\left[E_i\left(\frac{-r^2}{4\alpha_s t}\right) - E_i\left(-\lambda^2\right)\right] \quad 0 < r < S(t) \tag{2.a}$$

$$T_l(r,t) = T_i - \frac{\left(T_i - T_m\right)}{E_i\left(\frac{-\lambda^2\alpha_s}{\alpha_l}\right)}\left[E_i\left(\frac{-r^2}{4\alpha_s t}\right)\right] \quad S(t) < r < \infty \tag{2.b}$$

where the eigenvalues λ and the solidification front S (t) are given by

$$\frac{Q}{4\pi k_s}e^{-\lambda^2} + \frac{k_l\left(T_i - T_m\right)}{E_i\left(\frac{-\lambda^2\alpha_s}{\alpha_l}\right)}e^{\frac{-\lambda^2\alpha_s}{\alpha_l}} = -\lambda^2\alpha_s\rho L \tag{3.a}$$

$$S(t) = 2\lambda\sqrt{\alpha_s t} \tag{3.b}$$

In the above equations T_i is the uniform initial temperature, T_m is the melting temperature of the material, L is the latent heat of solidification of the material, ρ is the density, k_s and k_l are the thermal conductivities of the solid and liquid phases, respectively, α_s and α_l are the thermal diffusivities of the solid and liquid phases, respectively, and T_s and T_l are temperatures of the solid and liquid phases, respectively.

APPLICATIONS

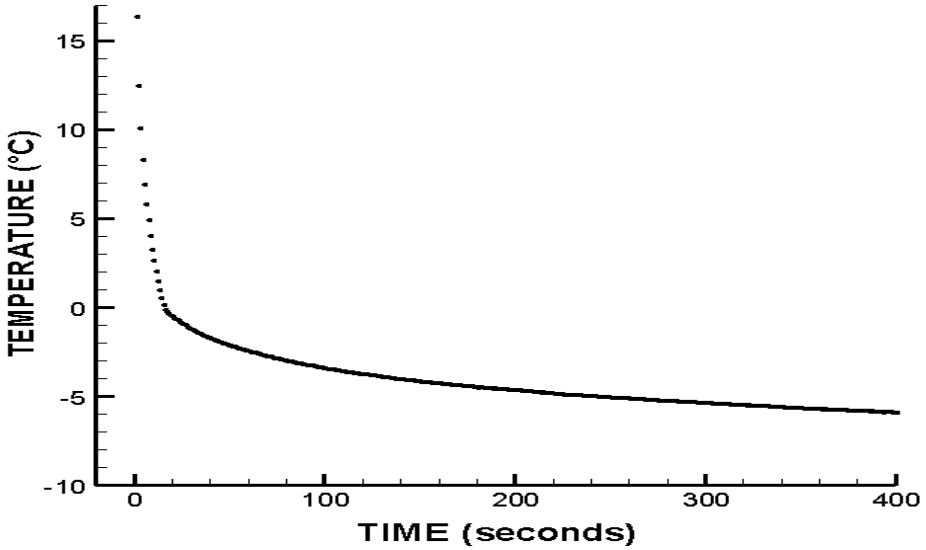
Estimation of the location of the solidification front and the intensity of a line heat sink

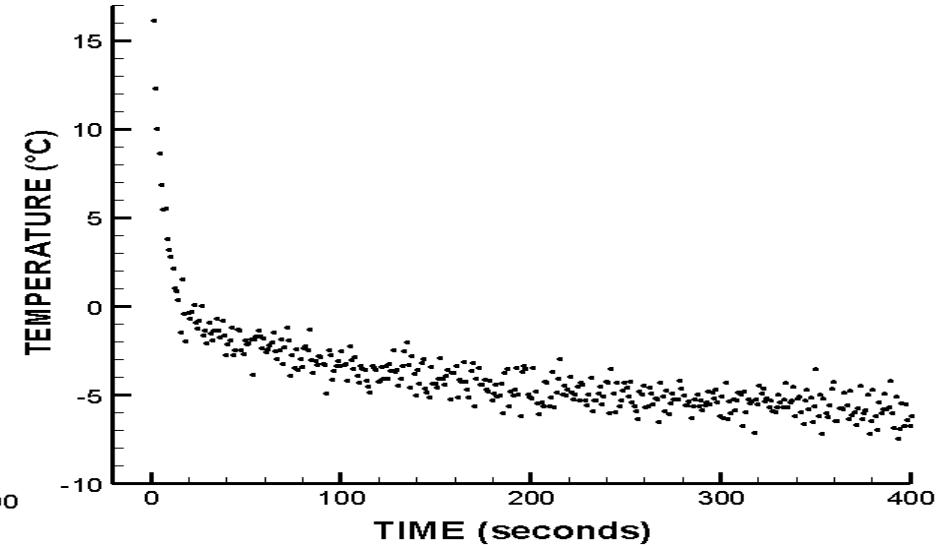
The physical problem defined by Eqs. (1.a-f) was solved analytically, where we used the following data, corresponding to solidifying water: $T_i = 25°C$, $T_m = 0°C$, $\alpha_s = 0.00118\frac{m^2}{s}$, $\alpha_l = 0.000146\frac{m^2}{s}$, $k_s = 2.22\frac{w}{m°c}$, $k_l = 0.61\frac{w}{m°c}$, $\rho = 997.1\frac{kg}{m^3}$, $L = 80\ \frac{\text{J}}{\text{kg}}$. The line heat sink was supposed to have a constant value equals to $\text{Q} = 50\frac{\text{W}}{\text{m}}$. In this work, the measurements (for the observation model) were obtained at *r*=0.01 m. The simulated noisy measurements were uncorrelated, additive, Gaussian, with zero mean and constant standard deviation equal to 5% of the maximum temperature. Figures 3.a,b show the transient measurements obtained after applying such constant line heat sink**,** with and without errors, respectively.

Silva, W. B. Orlande, H. R. B.; Colaço, M. J., Fudym, O., 2011, Application Of Bayesian Filters To A One-Dimensional Solidification Problem, *21st Brazilian Congress of Mechanical Engineering*, Natal, RN, Brazil.

APPLICATIONS

Estimation of the location of the solidification front and the intensity of a line heat sink





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Auxiliary Sampling Importance Resampling (ASIR) Algorithm

(Ristic, B., Arulampalam, S., Gordon, N., 2004, *Beyond the Kalman Filter*, Artech House, Boston)

Step 1
For i=1,...,N draw new particles $\mathbf{x}_k^i$ from the prior density $\pi(\mathbf{x}_k\|\mathbf{x}^i_{k-1})$ and then calculate some characterization of $\mathbf{x}_k$, given $\mathbf{x}^i_{k-1}$, as for example the mean μ^i_k=E[$\mathbf{x}_k\|\mathbf{x}^i_{k-1}$]. Then use the likelihood density to calculate the correspondent weights $w^i_k=\pi(\mathbf{z}_k\|\mu^i_k)w^i_{k-1}$
Step 2
Calculate the total weight $t=\Sigma_i\ w^i_k$ and then normalize the particle weights, that is, for i=1,...,N let $w^i_k = t^{-1}\ w^i_k$
Step 3
Resample the particles as follows :
Construct the cumulative sum of weights (CSW) by computing $c_i=c_{i-1}+w^i_k$ for i=1,...,N, with c_0=0
Let i=1and draw a starting point u_1 from the uniform distribution U[0,N^{-1}]
For j=1,...,N
Move along the CSW by making $u_j=u_1+N^{-1}(j-1)$
While $u_j>c_i$ make $i=i+1$ Assign sample $x^j_k=x^i_k$ Assign sample $w^j_k=N^{-1}$ Assign parent $i^j=i$
Step 4
For j=1,...,N draw particles $\mathbf{x}_k^j$ from the prior density $\pi(\mathbf{x}_k\|\mathbf{x}^{ij}_{k-1})$, using the parent i^j, and then use the likelihood density to calculate the correspondent weights $w^j_k=\pi(\mathbf{z}_k\|\mathbf{x}^j_k)\ /\ \pi(\mathbf{z}_k\|\mu^{ij}_k)$
Step 5
Calculate the total weight $t=\Sigma_j\ w^j_k$ and then normalize the particle weights, that is, for j=1,...,N let $w^j_k = t^{-1}\ w^j_k$

- The advantage of ASIR over SIR is that it naturally generates points from the sample at k-1, which, conditioned on the current measurement, are most likely to be close to the true state.
- The resampling is based on some point estimate μ^i_k that characterize $\pi(\mathbf{x}_k|\mathbf{x}^i_{k-1})$, which can be the mean μ^i_k=E[$\pi(\mathbf{x}_k|\mathbf{x}^i_{k-1})$] or simply a sample of $\pi(\mathbf{x}_k|\mathbf{x}^i_{k-1})$. If the state evolution model noise is small, $\pi(\mathbf{x}_k|\mathbf{x}^i_{k-1})$ is generally well characterized by μ^i_k, so that the weights w^i_k are more even and the ASIR algorithm is less sensitive to outliers than the SIR algorithm. On the other hand, if the state evolution model noise is large, the single point estimate μ^i_k in the state space may not characterize well $\pi(\mathbf{x}_k|\mathbf{x}^i_{k-1})$ and the ASIR algorithm may not be as effective as the SIR algorithm.

APPLICATIONS

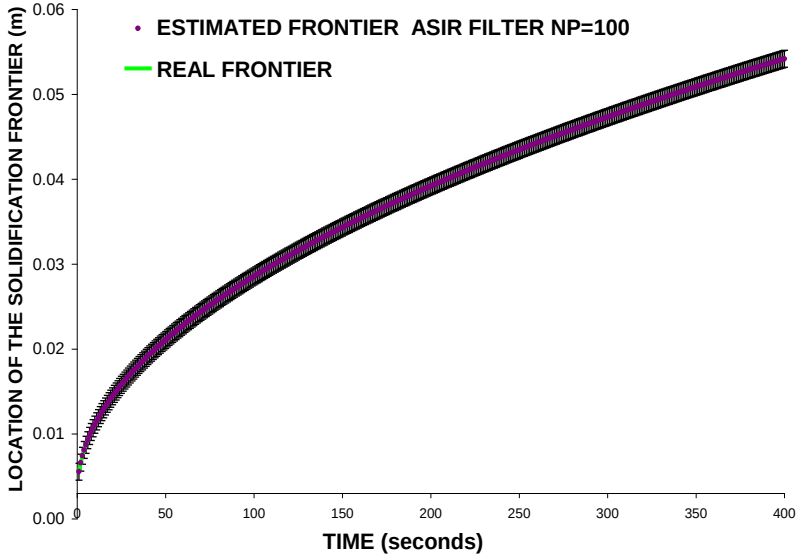
Estimation of the location of the solidification front and the intensity of a line heat sink

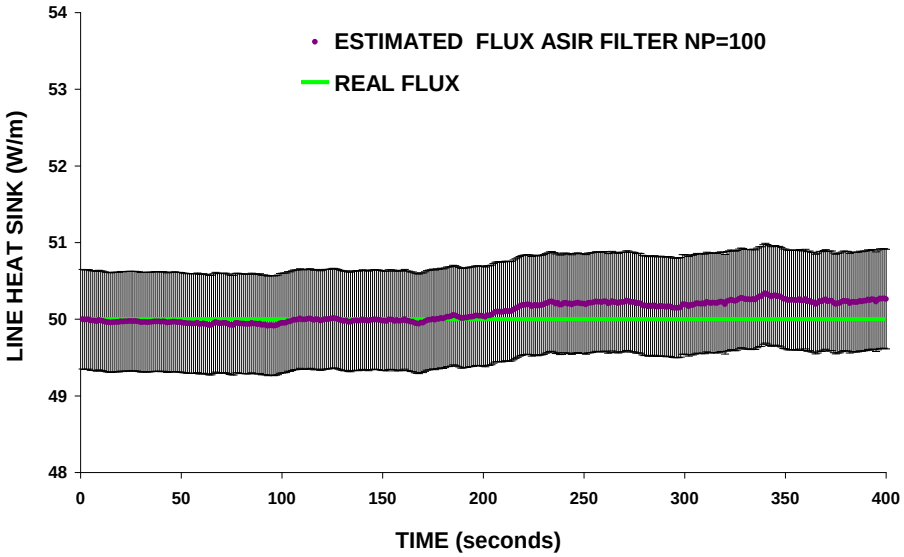
Bayesian filter	Number of Particles (NP)	Time	RMS error for the solidification front (m)	RMS error for the line heat sink intensity (W/m)
SIR	100	0.008 min.	$9x10^{-3}$	1.55
SIR	1000	0.997 min.	$2x10^{-3}$	1.78
SIR	5000	11.047 min.	$1x10^{-4}$	0.34
ASIR	100	0.161 min.	$7.9x10^{-5}$	0.15

Silva, W. B. Orlande, H. R. B.; Colaço, M. J., Fudym, O., 2011, Application Of Bayesian Filters To A One-Dimensional Solidification Problem, *21st Brazilian Congress of Mechanical Engineering*, Natal, RN, Brazil.

APPLICATIONS

Estimation of the location of the solidification front and the intensity of a line heat sink





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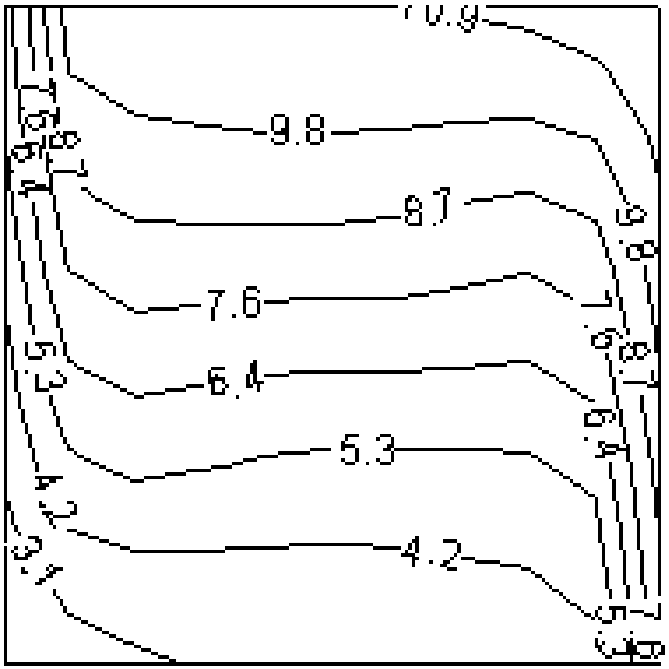
APPLICATIONS

Estimation of Unknown Heat Flux in Natural Convection

PHYSICAL PROBLEM

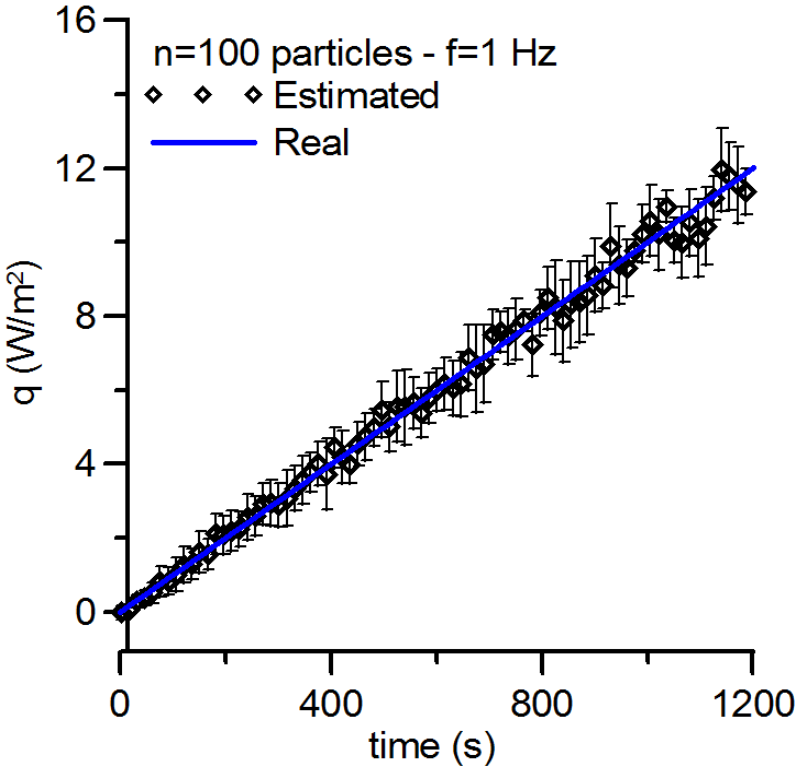
The physical problem under picture in this paper involves the transient laminar natural convection of a fluid inside a two-dimensional square cavity. The fluid is initially at rest and at the uniform temperature T_c. At time zero, the bottom and top surfaces are subjected to time-dependent heat fluxes $q_1(t)$ and $q_2(t)$, respectively. The left and right surfaces are subjected to constant temperatures T_c and T_h, respectively. The fluid properties are assumed constant, except for the density in the buoyancy term, where we consider Boussinesq's approximation valid.

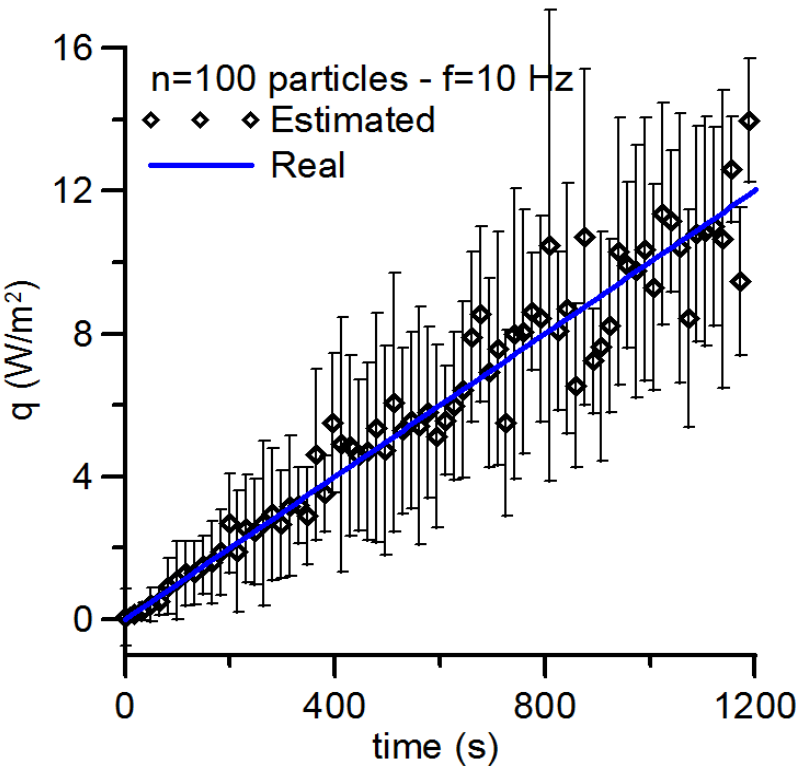


Colaço, M. J., Orlande, H. R. B.; Silva, W. B., Dulikravich, G., 2011, Application Of A Bayesian Filter To Estimate Unknown Heat Fluxes In A Natural Convection Problem, *ASME 2011 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference*, IDETC/CIE 2011, August 29-31, Washington, DC.

APPLICATIONS

Estimation of Unknown Heat Flux in Natural Convection





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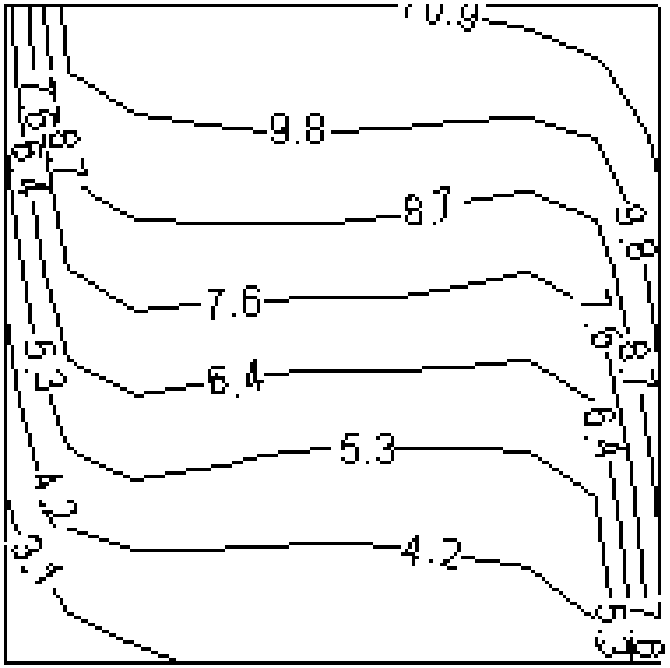
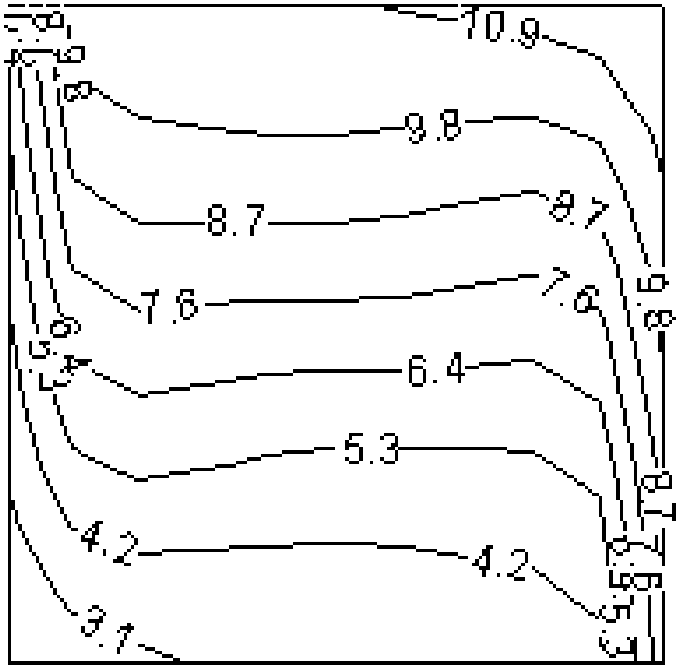
APPLICATIONS

Estimation of Unknown Heat Flux in Natural Convection

Exact



Estimated



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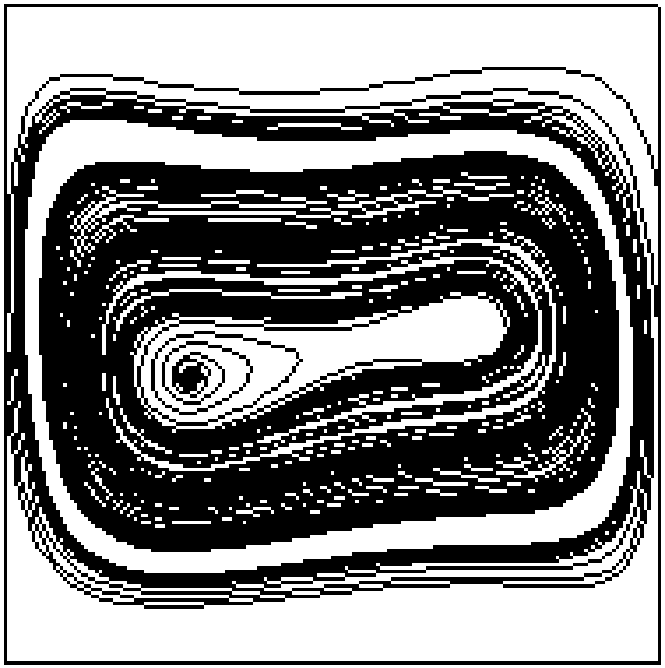
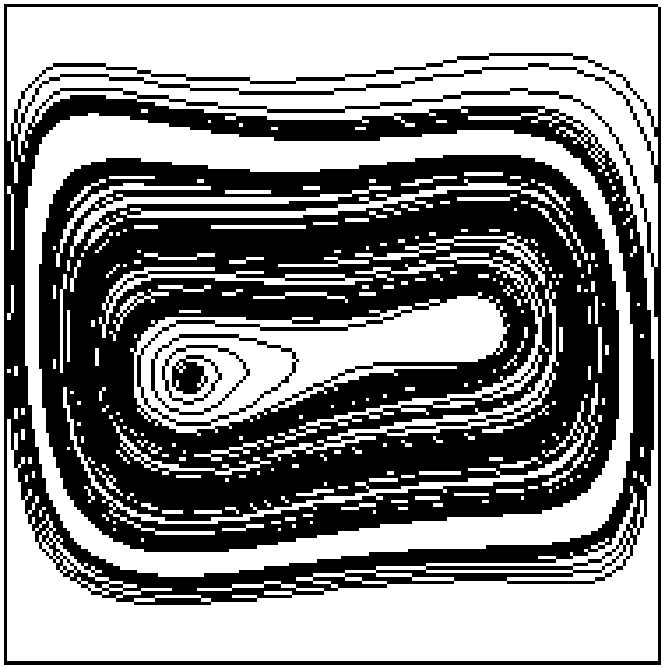
APPLICATIONS

Estimation of Unknown Heat Flux in Natural Convection

Exact



Estimated



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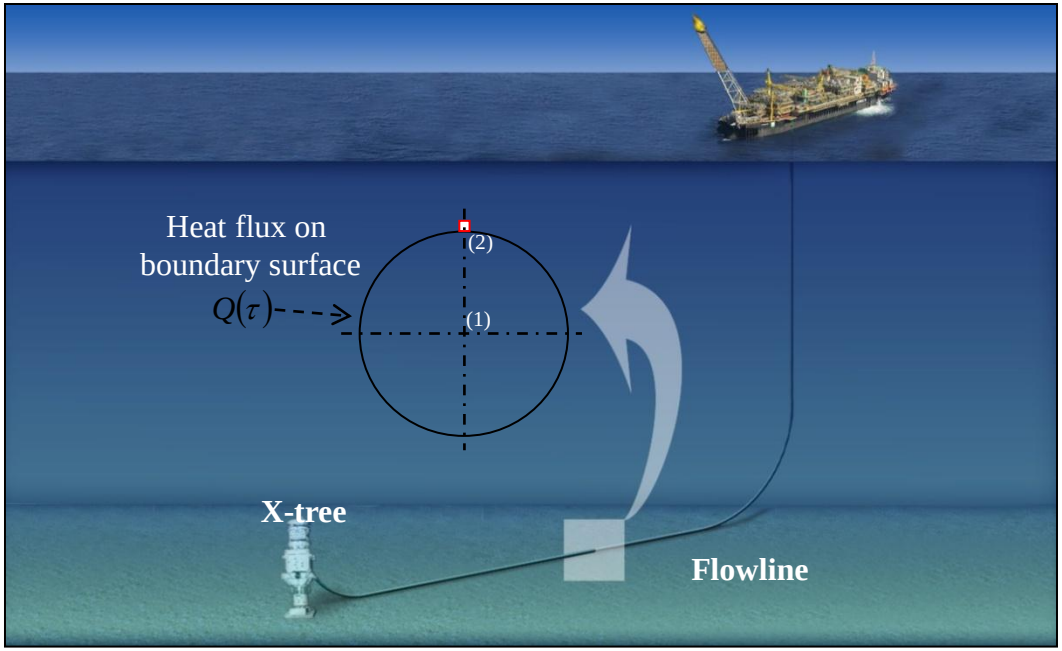
APPLICATIONS

Estimation and Control of the Temperature Field in Oil Pipelines

The physical problem examined in this work is based on a critical operational condition involving a pipeline shutdown situation, where the produced fluid is assumed stagnant. An optimal control approach was used to drive the predicted temperatures above a reference level.



- Measurement Point
- (1) R=0
- (2) R=1

Vianna, F., Orlande, H. R. B., Dulikravich, G., 2010, Optimal Heating Control To Prevent Solid Deposits In Pipelines, *V European Conference On Computational Fluid Dynamics* - ECCOMAS CFD 2010

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APPLICATIONS

The dimensionless mathematical formulation for this one-dimensional unsteady heat diffusion problem is given by

$$\frac{\partial\theta(R,\tau)}{\partial\tau}=\frac{\partial^2\theta(R,\tau)}{\partial R^2}+\frac{1}{R}\frac{\partial\theta(R,\tau)}{\partial R} \qquad 0\le R<1,\ \tau>0$$

$$\frac{\partial\theta(R,\tau)}{\partial R}+Bi\,\theta(R,\tau)=Q(\tau) \qquad R=1,\ \tau>0$$

$$\theta(R,0)=1 \qquad 0\le R<1,\ \tau=0$$

$\theta(R,\tau)$ is the dimensionless temperature distribution into the medium.

The fluid was considered as homogeneous, isotropic and with constant thermophysical properties.

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APPLICATIONS

• Control strategy is in accordance with the optimum control theory for linear problems.

• The aim of the associated optimal control problem is to find the control inputs $\boldsymbol{u}_k$ (heat flux on boundary surface) that minimizes the difference between the fluid temperature field and a desired profile .

$$\overline{u}_k = u_k - u^*$$

$$\overline{x}_k = x_k - x^*$$

Where $\boldsymbol{u}^*$ and $\boldsymbol{x}^*$ refer to the steady values of the control input and state variables

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APPLICATIONS

In terms of the linear quadratic regulator problem, the optimal values of the control input $\bar{u}_k$ are obtained by minimizing the following quadratic cost functional:

$$J = \int_{t_i}^{t_f} (x^T \mathbf{Q} x + u^T \mathbf{R} u)\, d\tau$$

where the weighting matrices **Q** and **R** are symmetric positive definite.

The solution to the optimal control problem is the state feedback control law:

$$\bar{u}_k = -\mathbf{K}\bar{x}_k$$

where the discrete-time state feedback gain **K** is of the form

$$\mathbf{K} = (\mathbf{R} + \mathbf{B}^T \mathbf{S} \mathbf{B})^{-1}\, \mathbf{B}^T \mathbf{S} \mathbf{A}$$

APPLICATIONS

Matrix **S** is the steady state solution to the discrete-time *Riccati* equation:

$$0 = \mathbf{A}^{\mathbf{T}}\mathbf{S}\mathbf{A} - \mathbf{S} + \mathbf{Q} - \mathbf{A}^{T}\mathbf{S}\mathbf{A}\mathbf{B}(\mathbf{R} + \mathbf{B}^{T}\mathbf{S}\mathbf{B})^{-1}\,\mathbf{B}^{T}\mathbf{S}\mathbf{A}$$

Thus, the control input u_k can be calculated from the control law as

$$\mathbf{u}_k = \mathbf{u}^* - \mathbf{K}(\mathbf{x}_k - \mathbf{x}^*)$$

However, when state variables are not directly available for control, an observer (KALMAN FILTER OR PARTICLE FILTER) was built to estimate the state variables from the input and output variables of the system.

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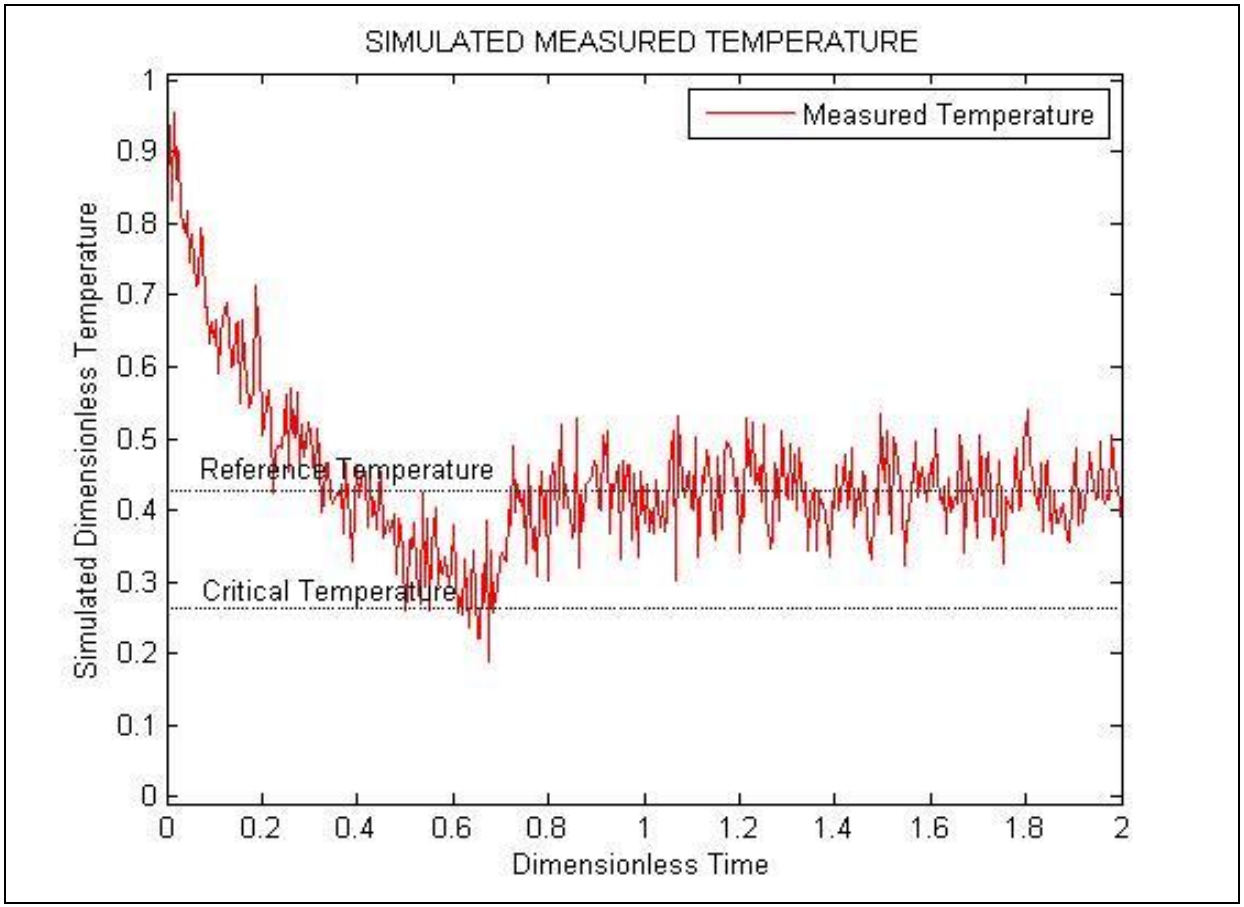
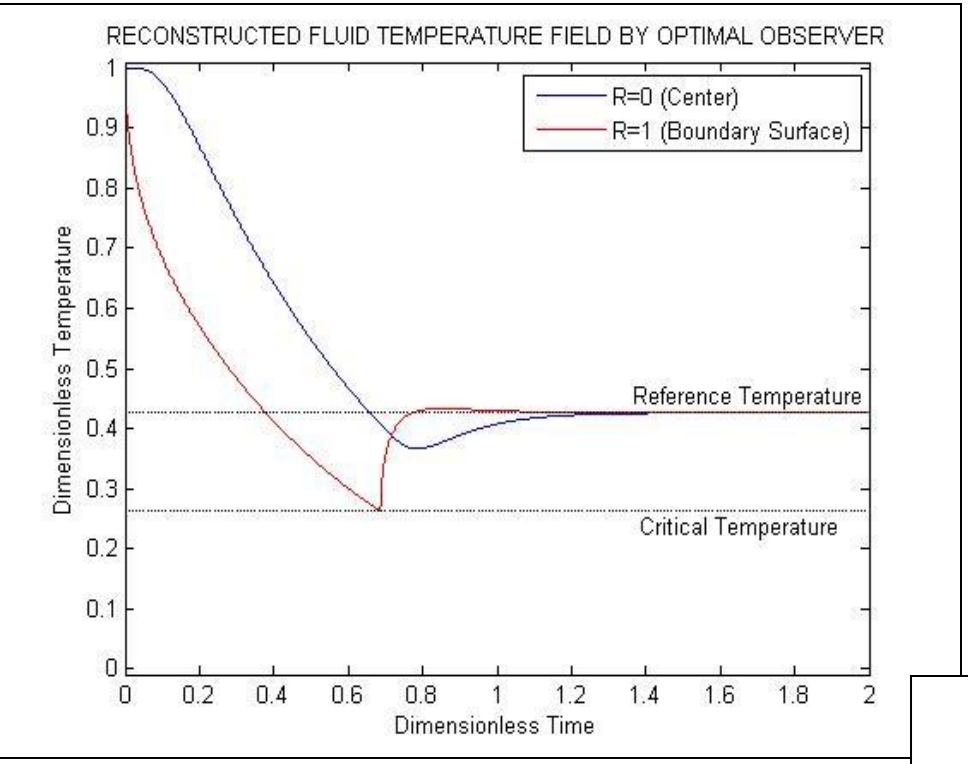
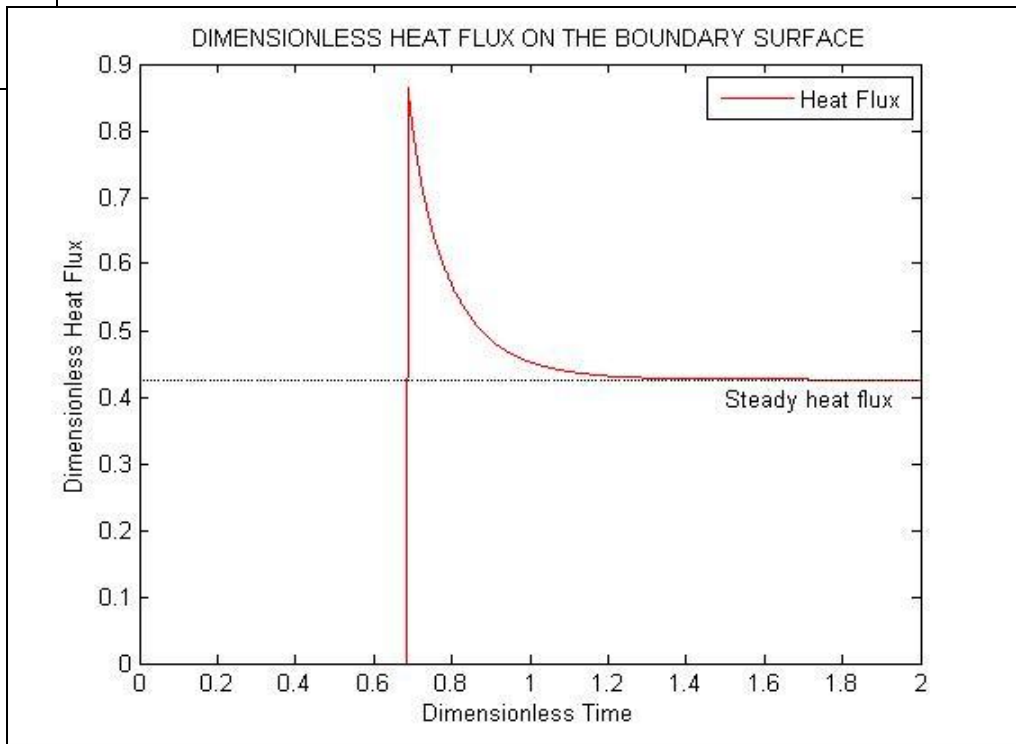
APPLICATIONS

Simulated measurements with standard deviation of 3 ^{o}C in the observation model error





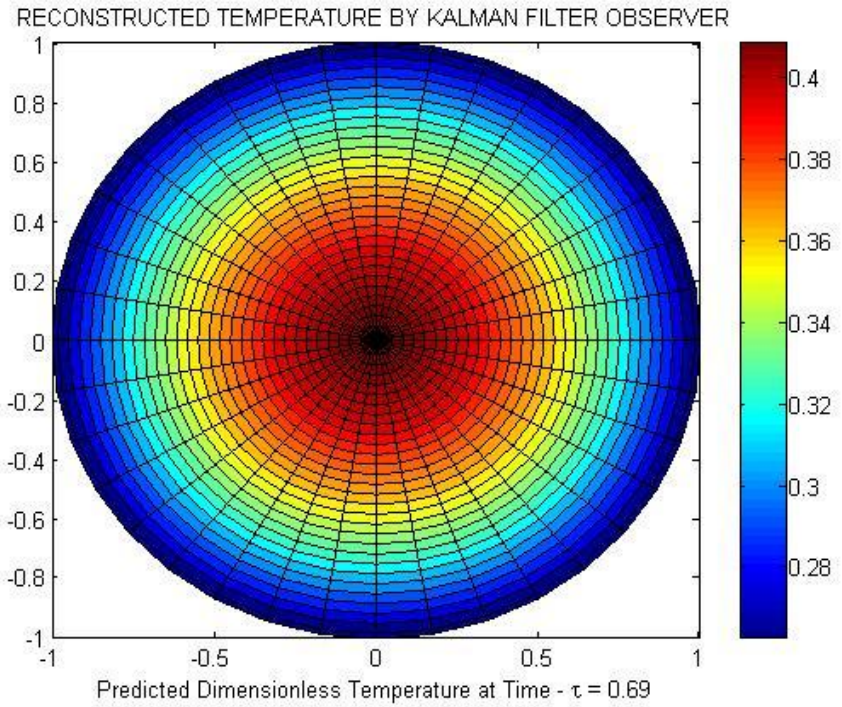
Evolution model with standard deviation of 0.01 ^{o}C
KALMAN FILTER

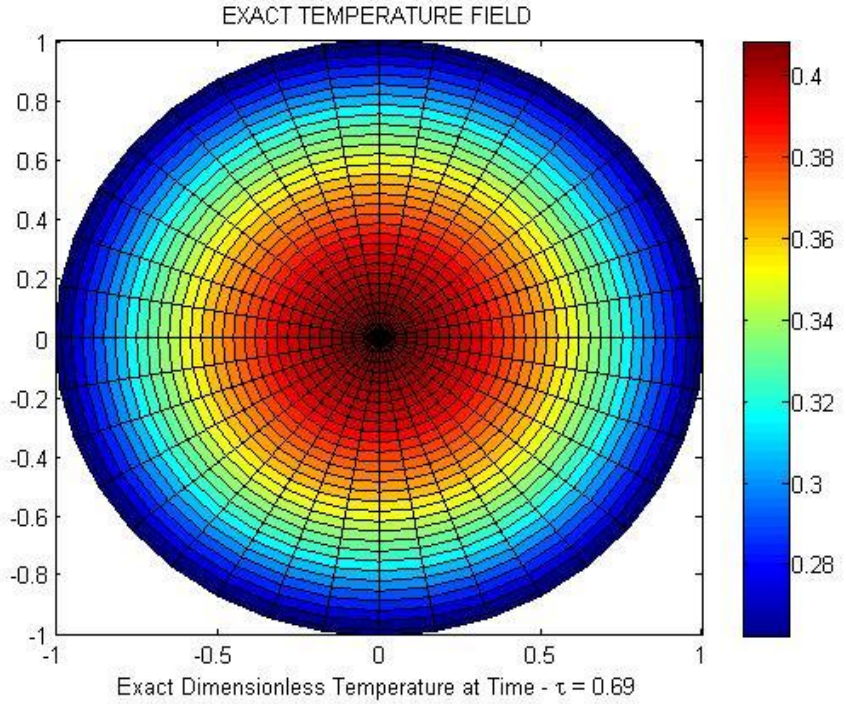


Temperature Predicted with the Kalman Filter and the Exact Temperature

Evolution model with standard deviation of 0.01 ^{o}C

$\tau = 0.69$

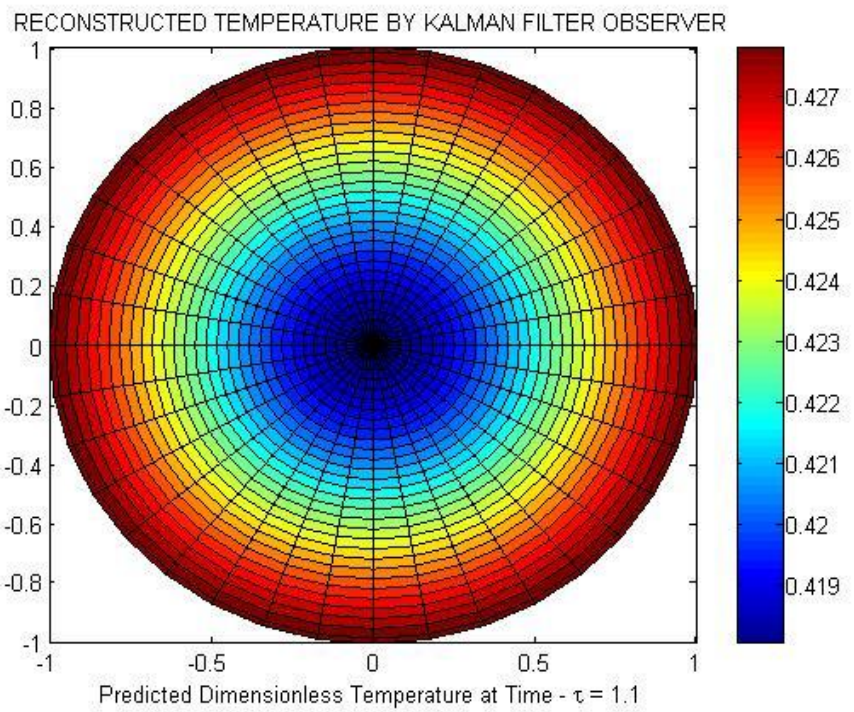


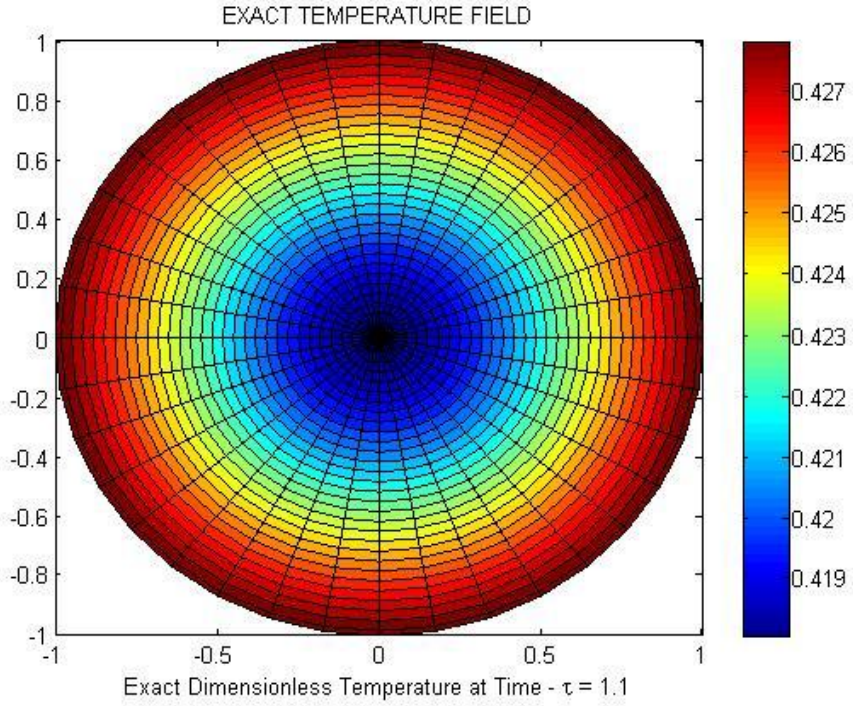


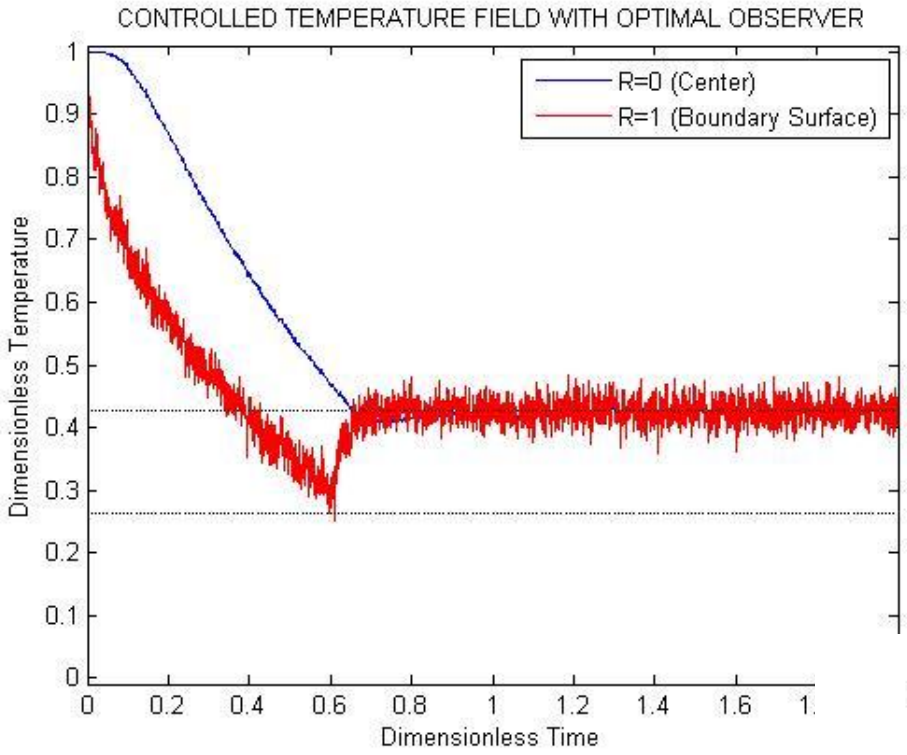
Temperature Predicted with the Kalman Filter and the Exact Temperature

Evolution model with standard deviation of 0.01 ^{o}C

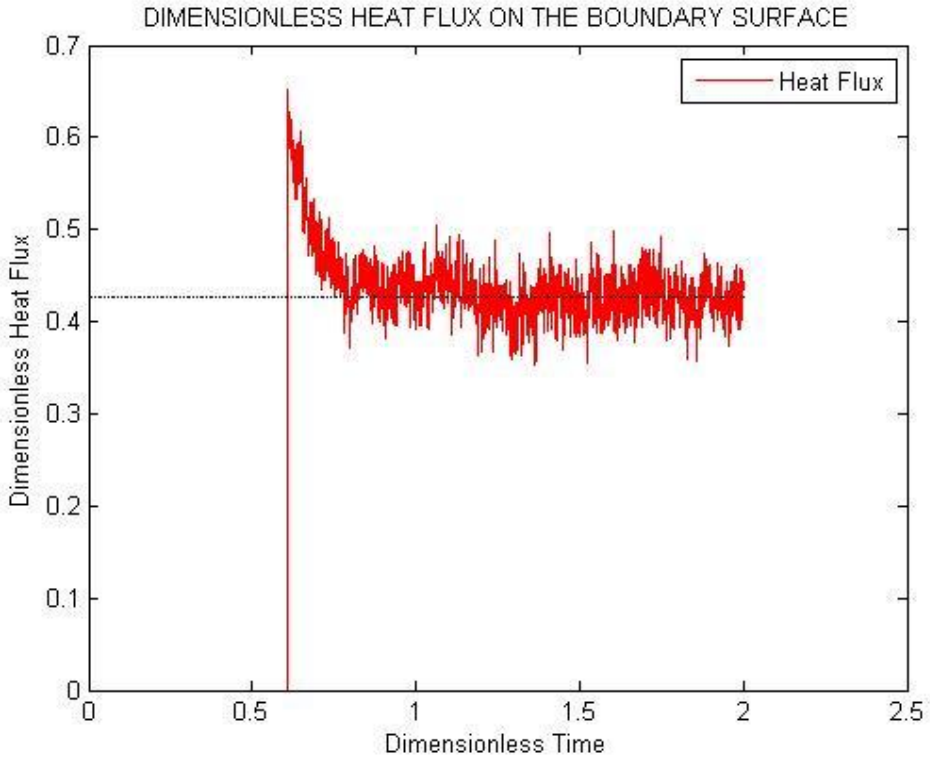
$\tau = 1.1$







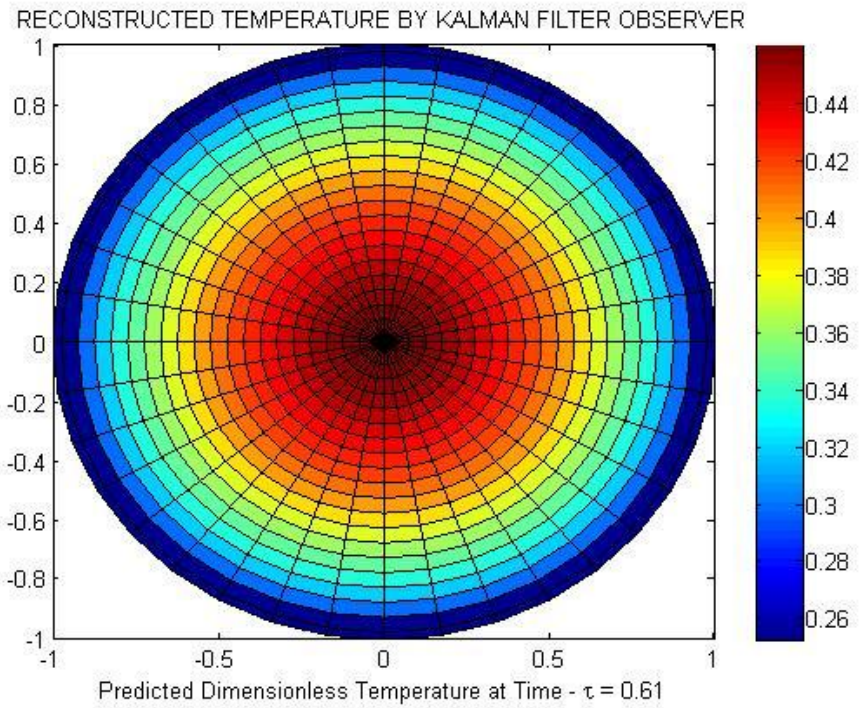
Evolution model with standard deviation of 1 ^{O}C
KALMAN FILTER

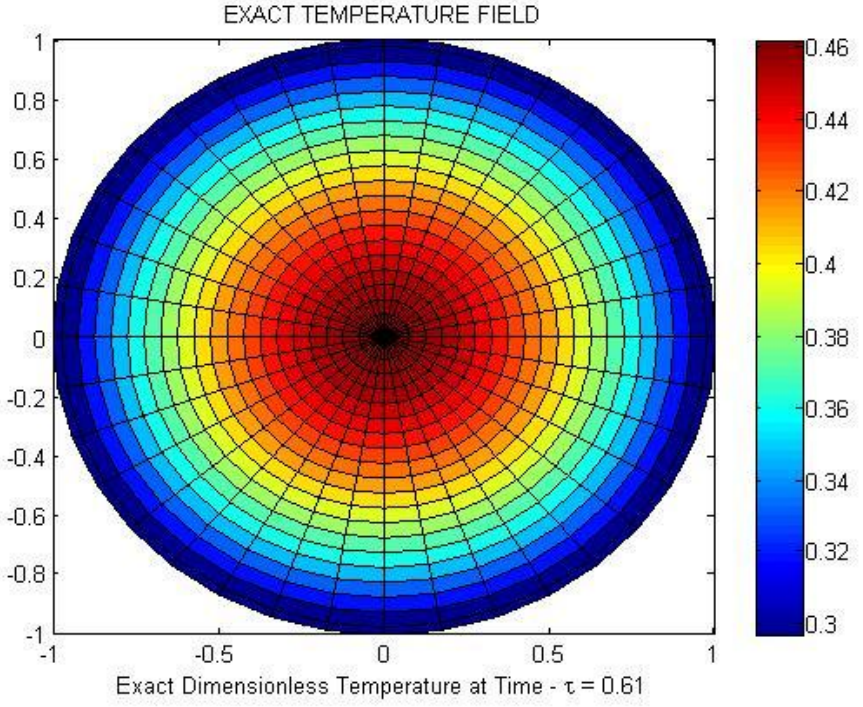


Temperature Predicted with the Kalman Filter and the Exact Temperature

Evolution model with standard deviation of 1 ^{o}C

$\tau = 0.61$

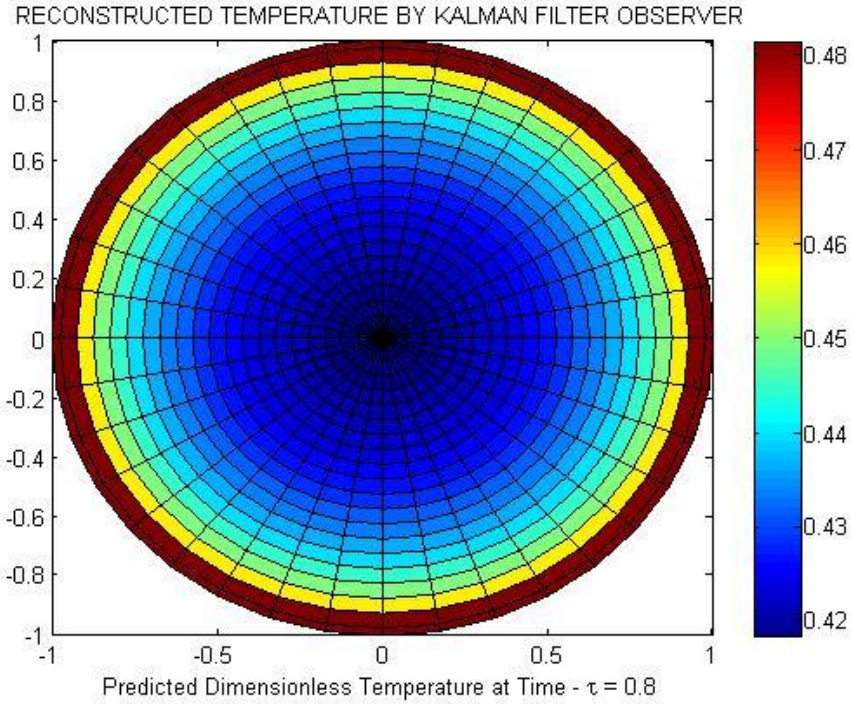


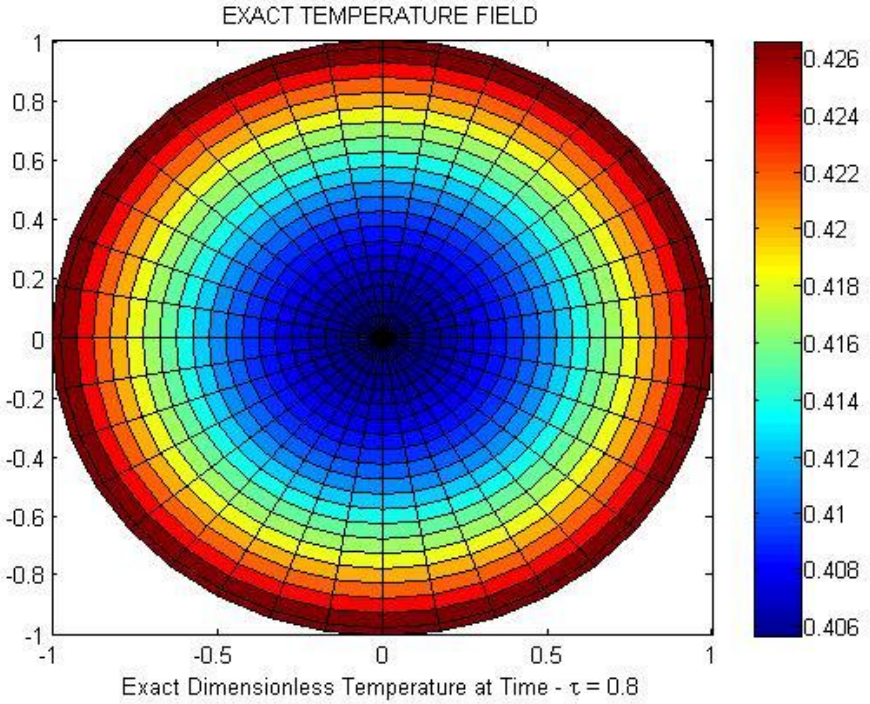


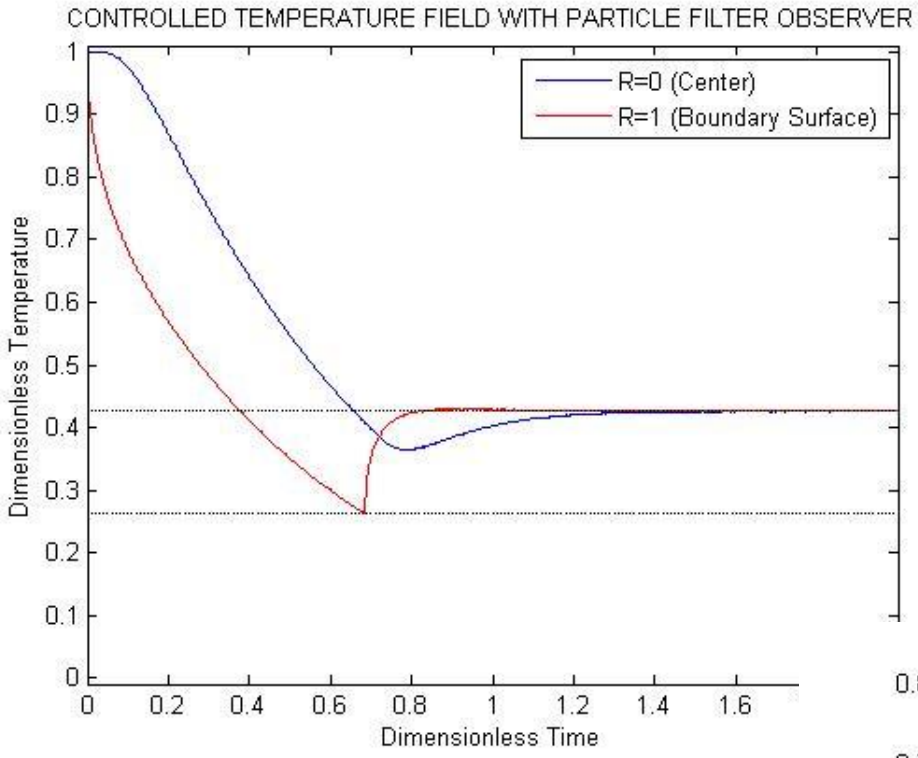
Temperature Predicted with the Kalman Filter and the Exact Temperature

Evolution model with standard deviation of 1 ^{o}C

$\tau = 0.8$

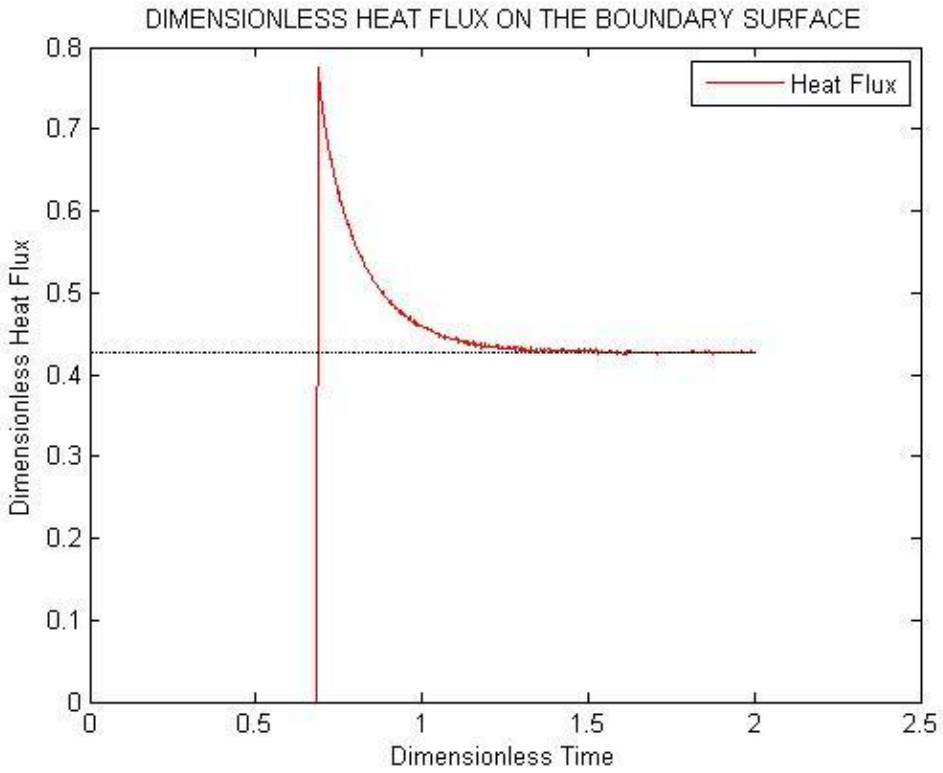






Evolution model with standard deviation of 1 ºC
PARTICLE FILTER

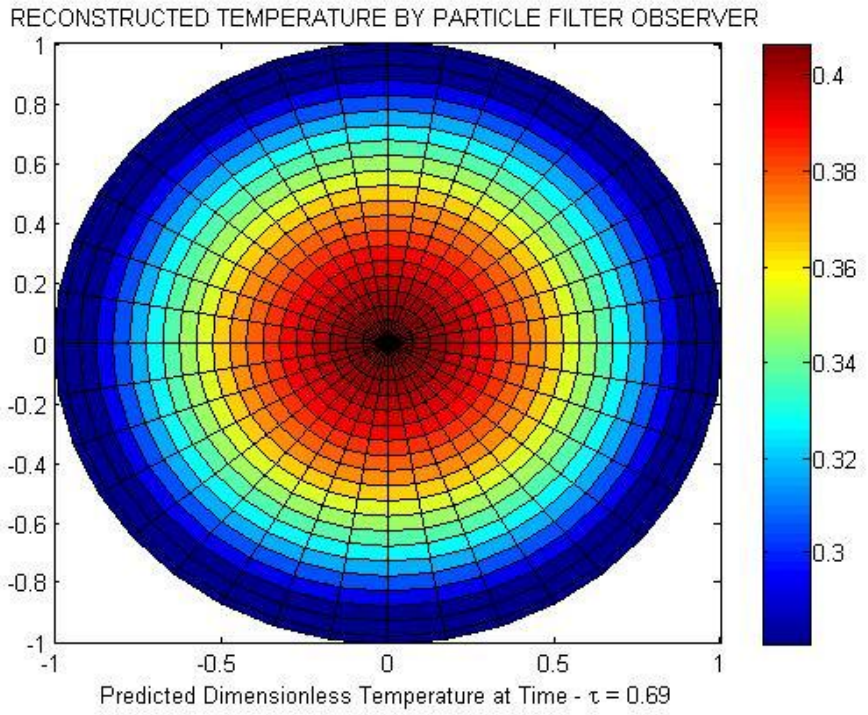
Number of Particles: 200

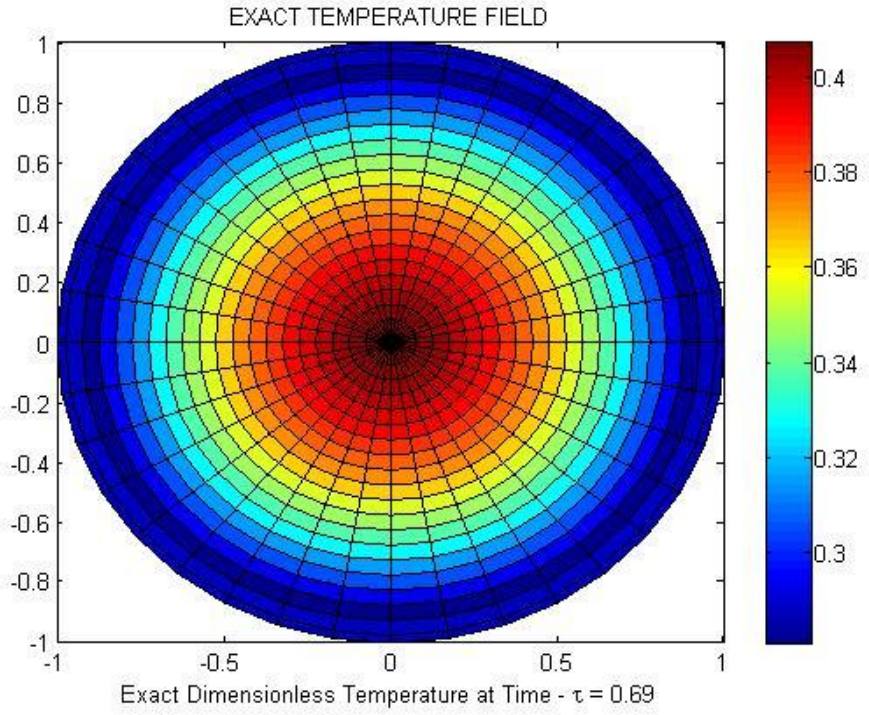


Temperature Predicted with the Particle Filter and the Exact Temperature

Evolution model with standard deviation of 1 ^{o}C

$\tau = 0.69$

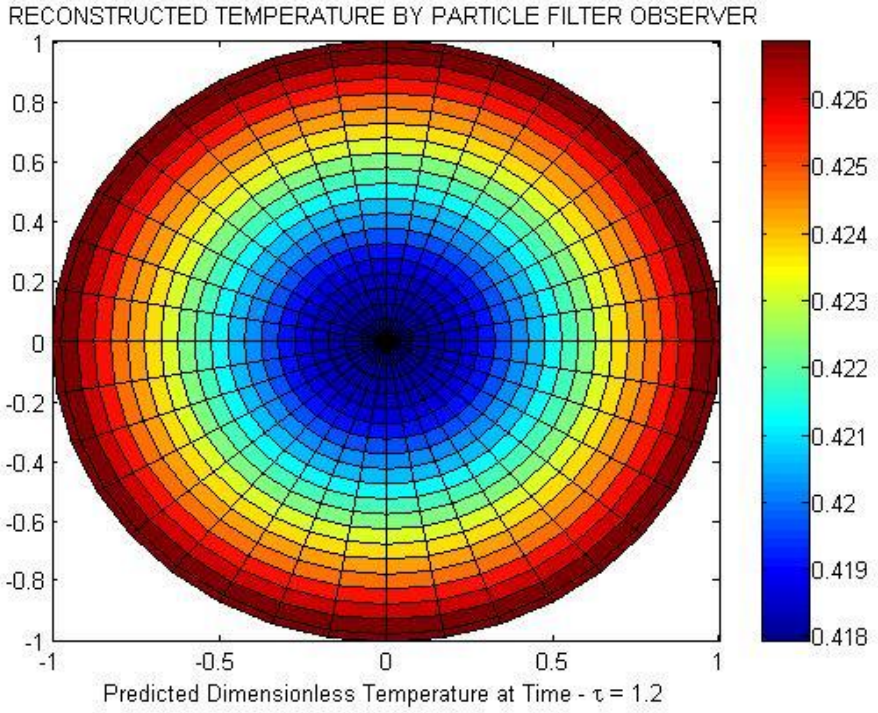


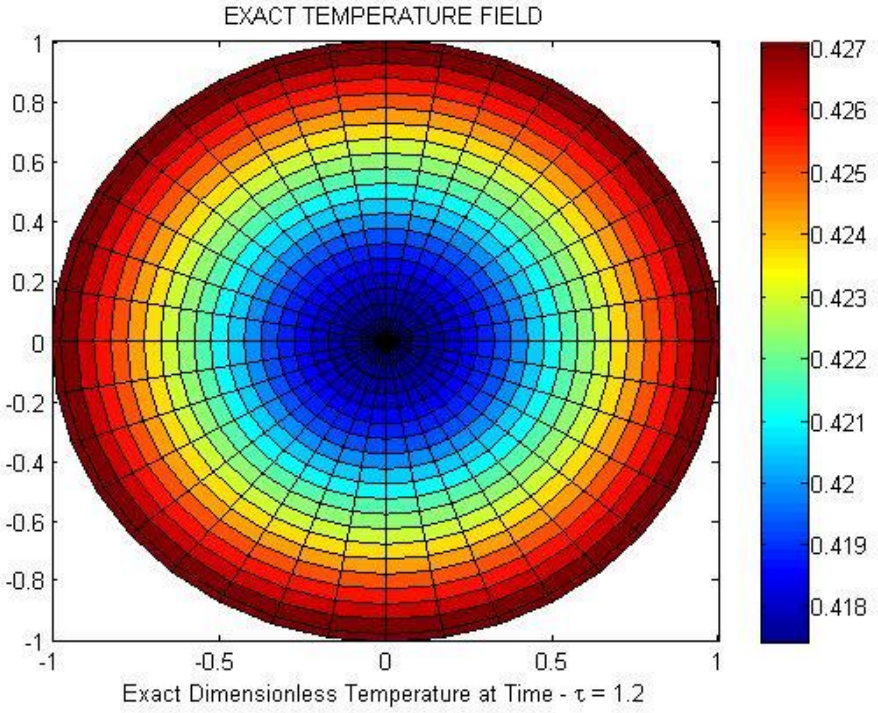


Temperature Predicted with the Particle Filter and the Exact Temperature

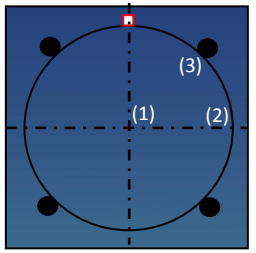
Evolution model with standard deviation of 1 ^{o}C

$\tau = 1.2$

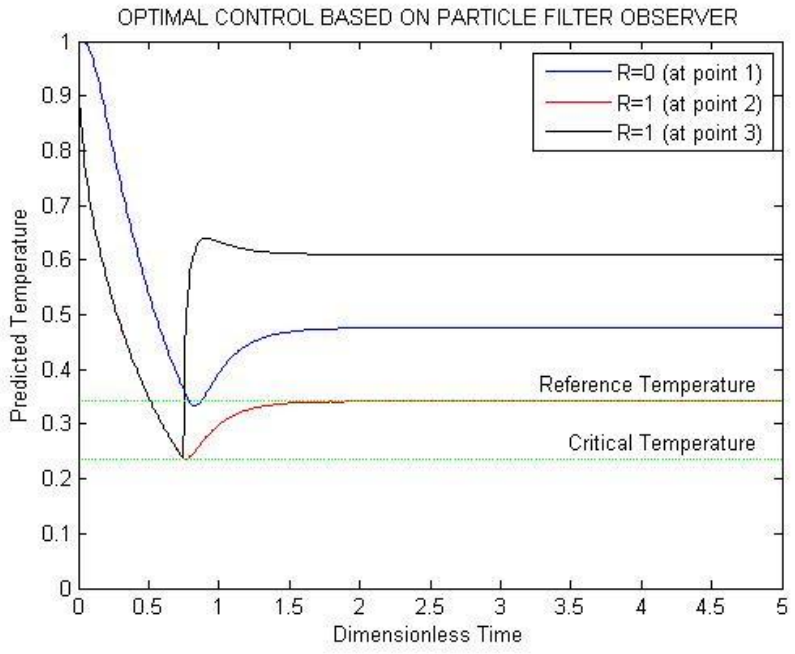




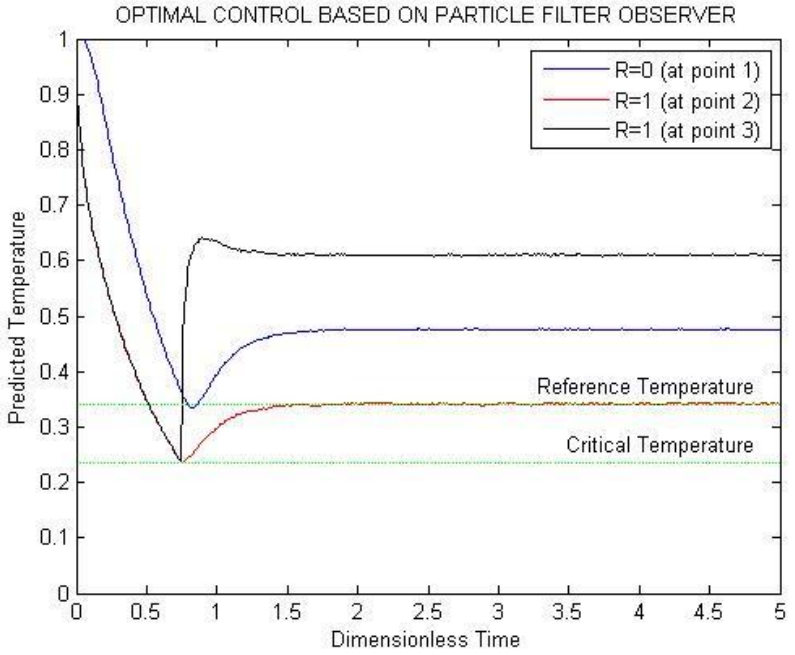
Two-dimensional case with Particle Filter Observer



Simulated measurements with standard deviation of 3 ^{o}C in the observation model error



Standard deviation of the evolution model error of 1^{o}C.



Standard deviation of the evolution model error of 3^{o}C.

CONCLUSIONS

- Kalman filter provides optimal solutions for linear-Gaussian evolution-observation models.
- Particle filter is the most general and robust technique for non-linear models and/or non-Gaussian distributions.
- ASIR algorithm is faster and requires less particles than the SIR algorithm.

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